

# PRIVILEGED COORDINATES AND TANGENT GROUPOID FOR CARNOT MANIFOLDS

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**ABSTRACT.** This paper is part of a series of papers on the construction of a full hypoelliptic pseudodifferential calculus on Carnot manifolds (a.k.a. filtered manifolds). A Carnot structure on a manifold is given by a flag of subbundles of the tangent bundle which is compatible with the Lie bracket of vector fields. Carnot manifolds include equiregular Carnot-Carathéodory manifolds, as well as some non-Carnot-Carathéodory manifolds that are naturally associated with equiregular Carnot-Carathéodory manifolds. The focus of this paper is on the infinitesimal structure of Carnot manifolds and the relevant notion of local coordinates in this setup. We produce a refinement of privileged coordinates for Carnot manifolds for which the extrinsic tangent group agree with the intrinsic tangent group. We call these coordinates *Carnot coordinates*. These coordinates have various applications and we present a number of them in this paper. This lead us to the second goal of this paper, which is to use these coordinates to establish an approximation result for maps between Carnot manifolds that are compatible with the Carnot manifold structures. Here the map is approximated in a suitable way by its tangent map, which is defined as a Lie group map between the corresponding tangent groups. This result is new and is the complete analogue in the setup of Carnot manifolds of the tangent linear approximation of smooth maps between manifolds. The third goal of this paper is to make use of the Carnot coordinates and their properties to construct an analogue for Carnot manifolds of the tangent groupoid of Connes. We then obtain a differentiable groupoid which encodes the smooth deformation of the pair groupoid  $M \times M$  to the tangent Lie group bundle of the Carnot manifold. The existence of such a groupoid was conjectured by Bellaïche. This shows that the tangent group of Carnot manifold at a given point is tangent in a true differential-geometric fashion. In particular, it improves a well-known result of Mitchell which establishes tangency in the sense of metric spaces.

## 1. INTRODUCTION

This paper is part of a series of papers whose goal is the construction of a full hypoelliptic pseudodifferential calculus on Carnot manifolds. More precisely, we seek for extending to Carnot manifolds the hypoelliptic pseudodifferential calculus on Heisenberg manifolds of Beals-Greiner [BG] and Taylor [Ta] and its subsequent developments (see, e.g., [BGS, Po3, Po2]). However, although there are a number of important related works (see, e.g., [CGGP, Cu, Me, RS]), the construction of such a pseudodifferential calculus is still an open question to date. Such a pseudodifferential calculus would have numerous potential applications to the analysis of hypoelliptic operators on Carnot manifolds and the global analysis of Carnot manifolds and its relationships with sub-Riemannian geometries. In particular, this should provide us with the tools for reformulating the index theorem of Atiyah-Singer [ASi1, ASi2] for hypoelliptic operators on Carnot manifolds. This should also prove useful to the analysis on asymptotically symmetric manifolds in the light of the program of Biquard-Mazzeo [BM].

We stress out that by a Carnot manifold we actually mean a manifold  $M$  together with a flag subbundles,

$$(1.1) \quad H_1 \subset H_2 \subset \cdots \subset H_r = TM,$$

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which is compatible with the Lie bracket of vector fields, i.e.,

$$[H_i, H_j] \subset H_{i+j} \quad \text{for } i + j \leq r.$$

We refer to Section 3 for a list of various examples of Carnot manifolds. Important geometric examples are Carnot structures arising from equiregular Carnot-Carathéodory distributions. For instance, such Carnot structures naturally appear in the framework of parabolic geometry [CS]. Recall also that, near a singular point, a Carnot-Carathéodory distribution can always be desingularized into an equiregular Carnot-Carathéodory distribution (see, e.g., [Be, RS, Go, Je2]). Moreover, as we point out at the end of Section 3, even for studying equiregular Carnot-Carathéodory structures we are naturally led to consider non-Carathéodory Carnot structures that are not generated by Carnot-Carathéodory distributions. Therefore, even from the point of view of Carnot-Carathéodory geometry, it is important to work in a general framework that includes non-Carathéodory Carnot structures.

It is a well-known idea of Elias Stein [St, RS] that homogeneous convolution operators on graded nilpotent Lie groups (more precisely Carnot-type groups) should be models for pseudo-differential operators on Carnot-Carathéodory manifolds. As it turns out, in order to implement this idea an important issue to deal with is getting a thorough understanding of the infinitesimal structure of a Carnot manifold. More precisely, at a given point of a Carnot manifold  $(M, H)$ , the relevant tangent structure is given by a graded nilpotent Lie group. There are two ways to obtain such a tangent group. First, it can be intrinsically defined from the grading associated with the filtration of the tangent space defined by the Carnot structure (see, e.g., [CS, Me]; see also Section 2). This actually provides us with a bundle of Carnot groups  $GM$ . Second, there is an extrinsic construction in terms of nilpotent approximations of vector fields in local coordinates (see, e.g., [Be]; see also Section 5). For the latter construction the nilpotent Lie group structure depends on the choice of the local coordinates. It can be shown that in suitable coordinates, called *privileged coordinates* [AS, Be, Go, He, RS], the extrinsic tangent group and the intrinsic tangent group have isomorphic Lie algebras, and hence are isomorphic (see also Proposition 5.24 on this point). However, even if the extrinsic tangent group in privileged coordinates is isomorphic to the intrinsic tangent group, its construction in such coordinates is still not canonical.

One goal of this paper is to introduce a refinement of the privileged coordinates, called *Carnot coordinates*, in which the extrinsic and intrinsic tangent groups agree. As we shall see in this paper, these coordinates have numerous properties. Furthermore, it is believed this should provide us with a relevant apparatus for constructing the hypoelliptic pseudodifferential calculus on Carnot manifolds alluded to at the beginning of the introduction.

Our approach to the construction of Carnot coordinates stems from the following. Given privileged coordinates at a point, we construct explicit polynomial coordinate changes that allow us to exhaust all Carnot-type groups that may occur as extrinsic tangent groups (Theorem 6.9). The upshot of this result is twofold. First, it shows that the construction of the extrinsic tangent group is definitely not canonical. Second, by specializing this construction to the intrinsic tangent group, we obtain an effective construction of a polynomial coordinate change that transforms privileged coordinates into Carnot coordinates (Theorem 7.4). Moreover, this change of coordinate is the unique polynomial change of coordinate of its type that transforms the given privileged coordinates into Carnot coordinates (*ibid.*).

The construction of privileged coordinates of Bellaïche [Be] for Carnot-Carathéodory manifolds can be easily extended to arbitrary Carnot manifolds (see Section 4). Combining this with the aforementioned conversion of privileged coordinates into Carnot coordinates, and given local coordinates near a point  $a$  of the manifold and a suitable type of tangent frame  $(X_1, \dots, X_n)$  called *H-frame*, we obtain an effective construction of a change of variables  $x \rightarrow \varepsilon_a^X(x)$  that produces Carnot coordinates (Theorem 7.8). We call the resulting Carnot coordinates *normal Carnot coordinates*. Carnot coordinates at a given point are not unique, but we show how to obtain all of them in terms of suitable perturbations of the normal Carnot coordinates (Theorem 9.1). The order of the perturbation is actually a “weighted” order depending on the components and variables that arise from dilations naturally associated with the flag (1.1) (see Section 8 for a more precise account on this).

As it turns out, the coordinate change  $x \rightarrow \varepsilon_a^X(x)$  is the unique polynomial coordinate change of its type that produces Carnot coordinates (Theorem 9.2). Furthermore, the effectiveness of its construction shows that, given local coordinates, the map  $(a, x) \rightarrow \varepsilon_a^X(x)$  is given by universal formulas in terms of the data  $(X_1, \dots, X_n)$  and depends on a minimal amount of jets of these data (see Proposition 10.1 for the precise statement). Incidentally, this map depends continuously on the  $H$ -frame  $(X_1, \dots, X_n)$  (see Proposition 10.4). In addition, in the case of a Carnot-type group the map  $(a, x) \rightarrow \varepsilon_a(x)$  is simply the map  $(a, x) \rightarrow a^{-1} \cdot y$ , i.e., where  $\cdot$  is the multiplication of the group (see Proposition 9.3). More generally, on an arbitrary Carnot manifold, in Carnot coordinates at a given point the normal Carnot map is well approximated by the product of the tangent group at the point (Proposition 12.3). Ultimately, the normal Carnot map provides us with a *smooth* deformation to the product of the tangent group (Proposition 12.4).

A further application of Carnot coordinates concerns the notion of tangent map in the setup of Carnot manifolds. A fundamental tool in differential geometry is the tangent linear map of a smooth map  $F : M \rightarrow M'$  between manifolds. In local coordinates near a point  $a \in M$  and its image  $a' = F(a)$ , the map  $F(x)$  is approximated by its tangent linear map  $F'(a) : TM(a) \rightarrow TM'(a')$  in the sense that

$$(1.2) \quad F(x) = F'(a)(x - a) + O(|x - a|^2) \quad \text{near } x = a.$$

It is a fundamental question to obtain the relevant analogous approximation for Carnot manifold maps, i.e., smooth maps between Carnot manifolds that are compatible with the Carnot structures (*cf.* Definition 2.8). In particular, such a result is needed to prove the invariance under change of coordinates of the pseudodifferential calculus on Carnot manifolds and to obtain an intrinsic definition of principal symbol for such a calculus (see [Po3]). In the setting of Carnot manifolds we seek for a tangent map obtained as a Carnot group morphism (i.e., a graded Lie group morphism) between tangent groups. Bellaïche [Be, Proposition 5.20] obtained an approximation result for Carnot diffeomorphisms in privileged coordinates, where the Carnot diffeomorphism is approximated by a graded Lie algebra isomorphism between the Lie algebras. However, the latter map does not have an intrinsic interpretation and it is not a Lie group morphism in general.

The second main goal of this paper is to make use of Carnot coordinates to obtain the relevant analogue of (1.2) in the setup of Carnot manifolds. As it turns out, given a Carnot manifold map, elementary linear algebra enables us to define an *intrinsic* Carnot group map between the *intrinsic* tangent groups (*cf.* Proposition 2.29). This is used to establish the functoriality of the intrinsic tangent group (Proposition 2.30). More importantly, we prove that, in *Carnot coordinates*, this tangent map provides us with the relevant approximation of the original Carnot manifold map (Theorem 11.5). In particular, this improves Bellaïche's approximation result. Once again the order of approximation is a weighted notion of order, and so the approximation holds in a stronger sense than in (1.2). We also stress out that it is paramount to work in Carnot coordinates, since the approximation result does not hold in general privileged coordinates (*cf.* Remark 11.6).

A further fundamental question is understanding to which extent the tangent group bundle approximates the Carnot manifold. For equiregular Carnot-Carathéodory manifolds Mitchell [Mi] proved that the tangent group is tangent to the Carnot manifold in the metric-space sense of Gromov [GLP, Gro]. This result was extended to the non-regular case by Bellaïche [Be] (note that in the singular case the tangent space at stake is only a homogeneous space). The approaches of Mitchell and Bellaïche both make use of privileged coordinates and the identification of the metric tangent space with the tangent group is only done through an isomorphism with the extrinsic tangent space. Bellaïche [Be, pp. 74–75] conjectured that, for equiregular Carnot-Carathéodory manifolds and more generally near any regular point of a Carnot-Carathéodory manifold, we could get an even stronger result by constructing a version for Carnot manifolds of the tangent groupoid of Connes [Co2]. Given a manifold  $M$ , its tangent groupoid is a manifold with boundary that encodes the smooth deformation of  $M \times M$  to the tangent space  $TM$ . Furthermore, the vector bundle structure of  $TM$  naturally arises from this deformation.

The third main goal of this paper is to associate with any Carnot manifold  $(M, H)$  a tangent groupoid obtained as the differentiable groupoid encoding a smooth deformation of  $M \times M$  to

the tangent group bundle  $GM$  (Theorem 13.21). We also establish the functoriality of this construction (Theorem 13.26). In the case of equiregular Carnot-Carathéodory manifolds, this proves Bellaïche's conjecture. The construction of the tangent groupoid of a Carnot manifold and the proof of its functoriality heavily rely on the use of Carnot coordinates and their properties, and more especially, the tangent map approximation of Carnot manifolds map alluded to above. The group bundle structure  $GM$  naturally arises from this deformation, and so this shows that the tangent group is tangent to the Carnot manifold in a true *differential-geometric* fashion. Incidentally, this provides us with a further conceptual explanation for why the tangent space is a Carnot-type group.

The tangent groupoid of a manifold was introduced by Connes [Co2] in his proof of the index theorem of Atiyah-Singer [ASi1, ASi2] (see also [Hi]). This approach to the index theorem was extended to contact manifolds by van Erp [vE1, vE2] (see also [BvE]). Along this line of research we thus can expect the tangent groupoid of a Carnot manifold to play an important role in the reformulation of the index theorem for hypoelliptic operators on Carnot manifolds. Note also there are close relationships between tangent groupoids and pseudodifferential calculi (see, e.g., [DS]). Therefore, at least conceptually, the tangent groupoid of a Carnot manifold should also prove useful to the construction of a hypoelliptic pseudodifferential calculus on Carnot manifolds.

This paper is organized as follows. In Section 2, we recall the main definitions concerning Carnot manifolds and the intrinsic construction of their tangent Lie group bundles. We include the definition of intrinsic tangent maps. In Section 3, we describe various examples of Carnot manifolds. As we are working in the setting of Carnot manifolds, we need to explain in some details how to adapt to this setting Bellaïche's construction of privileged coordinates. This is carried out in Section 4. In Section 5, we give a detailed account on the nilpotent approximation of vector fields and how it enables us to obtain an extrinsic tangent group. In particular, we give a very precise sense to the various asymptotics at stake. In Section 6, we exhibit all the Carnot-type groups that occur as extrinsic tangent groups at a given point. Section 7 is devoted to the construction of Carnot coordinates. In Section 8, we gather some results about the relevant notion of approximation of multi-valued maps in our setting. This is used in Section 9 to give a characterization of Carnot coordinates and explain how to get all of them. In Section 10, we give a closer look at normal Carnot coordinates and study their dependence on the  $H$ -frame. In Section 11, we show that a Carnot manifold map is well approximated by its tangent map and derive several applications of this result. In Section 12, we prove that the normal Carnot map provides us with a smooth deformation to the product of the tangent group bundle. In Section 13, we make an extensive use of the Carnot coordinates and their properties to construct the tangent groupoid of a Carnot manifold and establish the functoriality of this construction.

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## 2. CARNOT MANIFOLDS AND THEIR TANGENT GROUP BUNDLE

In this section, we mention the main definitions regarding Carnot manifolds and explain the construction of the associated (intrinsic) tangent Lie group bundle. We also include an account on the construction of tangent maps in this setting. This will show the functoriality of the construction of the intrinsic tangent group construction.

### 2.1. Carnot manifolds.

**Definition 2.1.** A *Carnot-type algebra* is a nilpotent Lie algebra  $\mathfrak{g}$  together with a grading,

$$(2.1) \quad \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_r,$$

in such a way that

$$(2.2) \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \text{ for } i+j \leq r \quad \text{and} \quad [\mathfrak{g}_i, \mathfrak{g}_j] = \{0\} \text{ for } i+j > r.$$

*Remark 2.2.* The condition (2.2) implies that  $\mathfrak{g}$  is step  $r$  nilpotent Lie algebra.

**Definition 2.3.** A Carnot-type group (or simply a Carnot group) is a simply connected nilpotent Lie group whose Lie algebra is a Carnot-type algebra.

*Remark 2.4.* By a Carnot group it is often meant a simply connected nilpotent Lie group whose Lie algebra has a grading (2.1) such that  $\mathfrak{g}_{j+1} = [\mathfrak{g}_j, \mathfrak{g}_1]$  for  $j = 1, \dots, r-1$  (see, e.g., [Pa, BLU]). It is convenient to slightly enlarge the definition of a Carnot group so as to include Abelian groups and products of Carnot groups with Abelian groups.

**Definition 2.5.** A Carnot manifold is a pair  $(M, H)$ , where  $M$  is a manifold and  $H = (H_1, \dots, H_r)$  is a finite flag of subbundles of  $TM$  such that

$$(2.3) \quad H_1 \subset H_2 \subset \dots \subset H_r = TM \quad \text{and} \quad [H_i, H_j] \subset H_{i+j} \text{ for } i+j \leq r.$$

The number  $r$  is called the *step* of the Carnot manifold  $(M, H)$ . The sequence  $(\text{rk } H_1, \dots, \text{rk } H_r)$  is called its *type*.

*Remark 2.6.* Carnot manifolds are called filtered manifolds in [CS]. We prefer to use the terminology Carnot manifold because, as we are going to see, in this framework the natural notion of tangent structure at a given point is that of a Carnot-type group.

*Remark 2.7.* We postpone to the next section the description of various examples of Carnot manifolds. In fact, the description of these examples requires the definition of the tangent Lie algebra bundle and tangent Lie group bundle given in this section.

**Definition 2.8.** Let  $(M, H)$  and  $(M', H')$  be Carnot manifolds of step  $r$ , so that  $H = (H_1, \dots, H_r)$  and  $H' = (H'_1, \dots, H'_r)$ .

(1) We say that a smooth map  $\phi : M \rightarrow M'$  is Carnot manifold map when, for  $j = 1, \dots, r$ ,

$$(2.4) \quad \phi'(x)X \in H'_j(\phi(x)) \quad \text{for all } (x, X) \in H_j.$$

(2) We say that a smooth map  $\phi : M \rightarrow M'$  is Carnot diffeomorphism when it is a diffeomorphism and a Carnot manifold map.

*Remark 2.9.* If  $\phi : M \rightarrow M'$  is a Carnot diffeomorphism, then the condition (2.4) exactly means that  $\phi_* H_j = H'_j$  for  $j = 1, \dots, r$ . Therefore, in this case, the Carnot manifold  $(M, H)$  and  $(M', H')$  must have same type and the inverse map  $\phi^{-1}$  too is a Carnot diffeomorphism.

From now on, we let  $(M^n, H)$  be an  $n$ -dimensional Carnot manifold of step  $r$ , so that  $H = (H_1, \dots, H_r)$ , where the subbundles  $H_j$  satisfy (2.3).

**Definition 2.10.** The weight sequence of  $(M, H)$  is the sequence  $w = (w_1, \dots, w_n)$  defined by

$$(2.5) \quad w_j = \min\{w \in \{1, \dots, r\}; j \leq \text{rk } H_w\}.$$

*Remark 2.11.* Two Carnot manifolds have same type if and only if they have same weight sequence.

**Definition 2.12.** An  $H$ -frame over an open  $V \subset M$  is a tangent frame  $(X_1, \dots, X_n)$  over  $U$  such that, for all  $w = 1, \dots, r$ , the vector fields  $X_j$ ,  $w_j = w$ , are sections of  $H_w$ .

*Remark 2.13.* If  $(X_1, \dots, X_n)$  is an  $H$ -frame near  $a$ , then, for all  $w = 1, \dots, r$ , the vector fields  $X_j$ ,  $w_j \leq w$ , form a local frame of  $H_w$  near  $a$ .

**Definition 2.14.** A local  $H$ -chart, is a local chart  $\kappa$  from an open  $V \subset M$  onto an open  $U \subset \mathbb{R}^n$  together with an  $H$ -frame over  $V$ .

**2.2. The tangent Lie algebra bundle.** The flag  $H = (H_1, \dots, H_r)$  has a natural grading defined as follows. For  $w = 1, \dots, r$ , set  $\mathfrak{g}_w M = H_w / H_{w-1}$  (with the convention that  $H_0 = \{0\}$ ), and define

$$(2.6) \quad \mathfrak{g}M := \mathfrak{g}_1 M \oplus \dots \oplus \mathfrak{g}_r M.$$

Given  $a \in M$  and  $X \in H_w(a)$ , we shall denote by  $\dot{X}$  its class in  $\mathfrak{g}_w M(a)$ . In particular, if  $(X_1, \dots, X_n)$  is an  $H$ -frame near  $a$ , then the classes  $\dot{X}_j(a)$ ,  $w_j = w$ , form a basis of  $\mathfrak{g}_w M(a)$ .

In what follows, we let  $w$  and  $w'$  be weights in  $\{1, \dots, r\}$  such that  $w + w' \leq r$ .

**Lemma 2.15** ([CS, Me]). *Given  $a \in M$  let  $X$  (resp.,  $Y$ ) be a local section of  $H_w$  (resp.,  $H_{w'}$ ) near  $a$  (which we regard as a vector field). Then the class of  $[X, Y](a)$  in  $\mathfrak{g}_{w+w'}M(a)$  depends only on the respective classes of  $X(a)$  and  $Y(a)$  in  $\mathfrak{g}_wM(a)$  and  $\mathfrak{g}_{w'}M(a)$ .*

*Proof.* Let  $(X_1, \dots, X_n)$  be an  $H$ -frame near  $a$ . Then  $\{X_j; w_j \leq w\}$  and  $\{X_j; w_j \leq w'\}$  are local frames near  $a$  of  $H_w$  and  $H_{w'}$ , respectively. Therefore, near  $x = a$  we may write

$$X = \sum_{w_j \leq w} b_j(x) X_j \quad \text{and} \quad Y = \sum_{w_k \leq w'} c_k(x) X_k,$$

where the  $b_j(x)$  and  $c_k(x)$  are smooth functions. Set  $X_{[w]} = \sum_{w_j=w} b_j(x) X_j$  and  $Y_{[w']} = \sum_{w_k=w'} c_k(x) X_k$ . Then

$$X = X_{[w]} + X' \quad \text{and} \quad Y = Y_{[w']} + Y',$$

where  $X'$  and  $Y'$  are sections of  $H_{w-1}$  and  $H_{w'-1}$ , respectively. In particular, the respective classes of  $X(a)$  and  $Y(a)$  in  $\mathfrak{g}_wM(a)$  and  $\mathfrak{g}_{w'}M(a)$  depend only on the coordinate vectors  $(b_j(a))_{w_j=w}$  and  $(c_k(a))_{w_k=w'}$ . In addition, we have

$$[X, Y] = [X_{[w]}, Y_{[w']}] + [X_{[w]}, Y'] + [X', Y].$$

As  $[X_{[w]}, Y']$  and  $[X', Y]$  are sections of  $H_{w+w'-1}$  we see that

$$[X, Y](a) = [X_{[w]}, Y_{[w']}](a) \mod H_{w+w'-1}(a).$$

We observe that  $[X_{[w]}, Y_{[w]}]$  is equal to

$$\sum_{\substack{w_j=w \\ w_k=w'}} [b_j X_j, c_k X_k] = \sum_{\substack{w_j=w \\ w_k=w'}} b_j c_k [X_j, X_k] + \sum_{\substack{w_j=w \\ w_k=w'}} (b_j X_j (c_k) X_k - c_k X_k (b_j) X_j).$$

As the vectors fields in the 2nd summand of the r.h.s. are sections of  $H_{w+w'-1}$ , we deduce that

$$[X, Y](a) = \sum_{\substack{w_j=w \\ w_k=w'}} b_j(a) c_k(a) [X_j, X_k](a) \mod H_{w+w'-1}(a).$$

Thus the class of  $[X, Y](a)$  in  $\mathfrak{g}_{w+w'}M(a)$  depends only on the coordinate vectors  $(b_j(a))_{w_j=a}$  and  $(c_k(a))_{w_k=w'}$ , and hence depends only on the respective classes of  $X(a)$  and  $Y(a)$  in  $\mathfrak{g}_wM(a)$  and  $\mathfrak{g}_{w'}M(a)$ . The proof is complete.  $\square$

Let  $a \in M$ . It follows from Lemma 2.15 that there is a unique bilinear map,

$$\mathcal{L}_{w,w'}(a) : \mathfrak{g}_wM(a) \times \mathfrak{g}_{w'}M(a) \longrightarrow \mathfrak{g}_{w+w'}M(a),$$

such that, for all sections  $X$  of  $H_w$  near  $a$  and sections  $Y$  of  $H_{w'}$  near  $a$ , we have

$$\mathcal{L}_{w,w'}(a)(X(a), Y(a)) = \text{class of } [X, Y](a) \text{ in } \mathfrak{g}_{w+w'}M(a).$$

We note that this definition implies that

$$(2.7) \quad \mathcal{L}_{w,w'}(a)(X, Y) = -\mathcal{L}_{w',w}(a)(Y, X) \quad \text{for all } X \in \mathfrak{g}_wM(a) \text{ and } Y \in \mathfrak{g}_{w'}M(a).$$

The collection of the bilinear maps  $\mathcal{L}_{w,w'}(a)$ ,  $a \in M$ , forms a bilinear bundle map

$$\mathcal{L}_{w,w'} : \mathfrak{g}_wM \times \mathfrak{g}_{w'}M \rightarrow \mathfrak{g}_{w+w'}M.$$

We then have the following result.

**Lemma 2.16.**  $\mathcal{L}_{w,w'}$  is a smooth bilinear bundle map.

*Proof.* Given  $a \in M$ , let  $(X_1, \dots, X_n)$  be an  $H$ -frame near  $a$ . We know that the sections  $\dot{X}_i$  with  $w_i = w$  (resp.,  $w_i = w'$ ,  $w_i = w + w'$ ) form a local frame of  $\mathfrak{g}_wM$  (resp.,  $\mathfrak{g}_{w'}M$ ,  $\mathfrak{g}_{w+w'}M$ ) near  $a$ . Moreover, the fact that  $[H_{w_i}, H_{w_j}] \subset H_{w_i+w_j}$  for  $w_i + w_j \leq r$  implies that, near  $x = a$ , there are smooth functions  $L_{ij}^k(x)$ ,  $w_k \leq w_i + w_j$ , such that

$$(2.8) \quad [X_i, X_j] = \sum_{w_k \leq w_i + w_j} L_{ij}^k(x) X_k.$$



Therefore, when  $w_i = w$  and  $w_j = w'$ , taking classes in  $\mathfrak{g}_{w+w'}M$  we get

$$(2.9) \quad \mathcal{L}_{w,w'}(x) \left( \dot{X}_i(x), \dot{X}_j(x) \right) = \sum_{w_k=w+w'} L_{ij}^k(x) \dot{X}_k(x) \quad \text{near } x = a.$$

As the coefficients  $L_{ij}^k(x)$  depend smoothly on  $x$  we deduce that  $\mathcal{L}_{w,w'}$  is a smooth bilinear bundle map near  $x = a$ . This proves the lemma.  $\square$

**Definition 2.17.** The bilinear bundle map  $[\cdot, \cdot] : \mathfrak{g}M \times \mathfrak{g}M \rightarrow \mathfrak{g}M$  is defined as follows. For  $a \in M$  and  $X_j \in \mathfrak{g}_{w_j}M(a)$ ,  $j = 1, 2$ , we set

$$(2.10) \quad [X_1, X_2](a) = \begin{cases} \mathcal{L}_{w_1, w_2}(a)(X_1, X_2) & \text{if } w_1 + w_2 \leq r, \\ 0 & \text{if } w_1 + w_2 > r. \end{cases}$$

**Lemma 2.18.** The bilinear bundle map  $[\cdot, \cdot]$  is a smooth field of Lie brackets on  $\mathfrak{g}M$  such that

$$(2.11) \quad [\mathfrak{g}_w M, \mathfrak{g}_{w'} M] \subset \mathfrak{g}_{w+w'} M \quad \text{if } w + w' \leq r \quad \text{and} \quad [\mathfrak{g}_w M, \mathfrak{g}_{w'} M] = \{0\} \quad \text{if } w + w' > r.$$

*Proof.* As the restriction of  $[\cdot, \cdot]$  on  $\mathfrak{g}_{w_1}M \times \mathfrak{g}_{w_2}M$  either agrees with  $\mathcal{L}_{w_1, w_2}$  if  $w + w' \leq r$  or is zero if  $w_1 + w_2 > r$ , it follows from Lemma 2.16 that  $[\cdot, \cdot]$  is a smooth bilinear bundle map. Moreover, it follows from (2.7) that  $[\cdot, \cdot]$  is antisymmetric. Therefore, it only remains to check that, for any  $a \in M$ , the bilinear map  $[\cdot, \cdot](a)$  satisfies Jacobi's identity on  $\mathfrak{g}M(a)$ .

For  $i = 1, 2, 3$ , let  $X_i \in \mathfrak{g}_{w_i}M(a)$ . If  $w_1 + w_2 + w_3 > r$ , then all three brackets  $[X_1, [X_2, X_3]]$ ,  $[X_1, [X_2, X_3]]$  and  $[X_1, [X_2, X_3]]$  vanish, and hence trivially satisfy Jacobi's identity. Assume that  $w_1 + w_2 + w_3 \leq r$ . For  $i = 1, 2, 3$  let  $\tilde{X}_i$  be a section of  $H_{w_i}$  near  $a$  such that  $\tilde{X}_i(a)$  represents  $X_i$  in  $\mathfrak{g}_{w_i}M(a)$ . By definition each bracket  $[X_i, X_j](a)$  is represented by  $[\tilde{X}_i, \tilde{X}_j](a)$  in  $\mathfrak{g}_{w_i+w_j}M(a)$ , and so each two-fold bracket  $[X_i, [X_j, X_k]](a)$  is represented by  $[\tilde{X}_i, [\tilde{X}_j, \tilde{X}_k]](a)$  in  $\mathfrak{g}_{w_i+w_j+w_k}M(a)$ . Therefore, the Jacobi's identity for vector fields implies that

$$[X_1, [X_2, X_3]](a) + [X_2, [X_3, X_1]](a) + [X_3, [X_1, X_2]](a) = 0.$$

This shows that  $[\cdot, \cdot](a)$  satisfies Jacobi's identity on  $\mathfrak{g}M(a)$ . The proof is complete.  $\square$

**Remark 2.19.** We recursively define the commutator vector bundles  $\mathfrak{g}^{[w]}M$ ,  $w = 1, 2, \dots$ , by setting

$$\mathfrak{g}^{[1]}M = \mathfrak{g}M \quad \text{and} \quad \mathfrak{g}^{[w]}M = [\mathfrak{g}M, \mathfrak{g}^{[w-1]}M] \quad \text{for } w \geq 2.$$

Then Lemma 2.18 implies that

$$\mathfrak{g}^{[w]}M \subset \mathfrak{g}_w M \quad \text{if } w \leq r \quad \text{and} \quad \mathfrak{g}^{[w]}M = \{0\} \quad \text{if } w > r.$$

In particular, this shows that each Lie algebra  $\mathfrak{g}M(a)$  is a step  $r$  nilpotent Lie algebra.

Combining Lemma 2.18 and Remark 2.19 gives the following result.

**Proposition 2.20.**  $(\mathfrak{g}M, [\cdot, \cdot])$  is a smooth bundle of step  $r$  Carnot type Lie algebras.

**Definition 2.21.** The Lie algebra bundle  $(\mathfrak{g}M, [\cdot, \cdot])$  is called the *tangent Lie algebra bundle* of  $(M, H)$ .

**Remark 2.22.** Let  $(X_1, \dots, X_n)$  be an  $H$ -frame near a point  $a \in M$ . For  $j = 1, \dots, n$  let us denote by  $\dot{X}_j$  the class of  $X_j$  in  $\mathfrak{g}_{w_j}M$ . Then  $(\dot{X}_1, \dots, \dot{X}_n)$  is a local frame of  $\mathfrak{g}M$  near  $x = a$ . The structure constants of  $\mathfrak{g}M$  with respect to this frame are computed as follows. As in (2.8), there are unique smooth functions  $L_{ij}^k(x)$ ,  $w_k \leq w_i + w_j$ , such that

$$[X_i, X_j] = \sum_{w_k \leq w_i + w_j} L_{ij}^k(x) X_k.$$

Then using (2.9) and (2.10) we get

$$(2.12) \quad [\dot{X}_i, \dot{X}_j] = \begin{cases} \sum_{w_k=w_i+w_j} L_{ij}^k(x) \dot{X}_k & \text{if } w_i + w_j \leq r, \\ 0 & \text{if } w_i + w_j > r. \end{cases}$$

**2.3. The tangent Lie group bundle.** The nilpotent Lie algebra bundle  $\mathfrak{g}M$  is the Lie algebra bundle of a nilpotent Lie group bundle  $GM$  defined as follows. Given  $a \in M$  the Lie group structure on  $GM(a)$  is obtained by taking the exponential map  $\exp_a : \mathfrak{g}M(a) \rightarrow GM(a)$  to be the identity and using the Baker-Campbell-Hausdorff formula to define the product law on  $GM(a)$ . More explicitly, for  $X \in \mathfrak{g}M(a)$ , let  $\text{ad}_X : \mathfrak{g}M(a) \rightarrow \mathfrak{g}M(a)$  be the adjoint endomorphism of  $X$ , i.e.,

$$(2.13) \quad \text{ad}_X(Y) = [X, Y](a) \quad \text{for all } Y \in \mathfrak{g}M(a).$$

Note that if  $X \in \mathfrak{g}_w M(a)$ , then  $\text{ad}_X$  maps  $\mathfrak{g}_{w'} M(a)$  to  $\mathfrak{g}_{w+w'} M(a)$  if  $w + w' \leq r$  and vanishes on  $\mathfrak{g}_{w'} M(a)$  if  $w + w' > r$ . Thus,  $\text{ad}_X$  is a nilpotent endomorphism of  $\mathfrak{g}M(a)$ . Let us denote by  $\text{Der}(\mathfrak{g}M(a))$  the algebra generated by the adjoint endomorphisms  $\text{ad}_X$ ,  $X \in \mathfrak{g}M(a)$ . Then, any  $A \in \text{Der}(\mathfrak{g}_a M)$  maps  $\mathfrak{g}_w M(a)$  to  $\mathfrak{g}_{w+1} M(a)$  for all  $w < r$  and vanishes on  $\mathfrak{g}_r M(a)$ , so that  $A$  is a nilpotent endomorphism of  $\mathfrak{g}M(a)$ . Therefore, given any power series  $f(z) = \sum_{k \geq 0} a_k z^k$ ,  $a_k \in \mathbb{C}$ , we may define

$$f(A) := \sum_{k \geq 0} a_k A^k = \sum_{0 \leq k \leq r} a_k A^k.$$

In addition, we set

$$(2.14) \quad \phi(z) = (z + 1) \frac{\log(1 + z)}{z} = 1 - \sum_{k \geq 1} \frac{(-1)^k}{k(k+1)} z^k.$$

Bearing this in mind, given  $X$  and  $Y$  in  $GM(a)$ , the Baker-Campbell-Hausdorff formula gives a formula for the product of  $X$  and  $Y$ . Namely, we have

$$(2.15) \quad \begin{aligned} X \cdot Y &= X + \left( \int_0^1 \Phi(e^{\text{ad}_X} e^{s \text{ad}_Y} - I) ds \right) Y, \\ &= X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \dots \end{aligned}$$

It follows from (2.15) and the smoothness of the Lie bracket of  $\mathfrak{g}M$  that the above formula defines a smooth family of products  $GM(a) \times GM(a) \rightarrow GM(a)$ . As the grading (2.6) satisfies (2.11) we then arrive at the following statement.

**Proposition 2.23.**  *$GM$  is a smooth bundle of step  $r$  Carnot-type groups.*

**Definition 2.24.**  $GM$  is called the tangent Lie group bundle of  $(M, H)$ .

We shall now look at some basic properties of the tangent group bundle  $GM$ .

**Proposition 2.25.** *Let  $X \in GM(a)$ . Then  $X^{-1} = -X$ .*

*Proof.* As  $\text{ad}_X X = 0$ , we see that  $f(\text{ad}_X)(-X) = f(0)$  for any power series  $f(z)$ . Bearing this in mind we have

$$X \cdot (-X) = X + \int_0^1 \Phi(e^{(1-s)\text{ad}_X} - I)(-X) ds = X + \int_0^1 (-X) ds = 0.$$

Likewise  $(-X) \cdot X = 0$ . Therefore  $-X$  is the inverse of  $X$ . The result is thus proved.  $\square$

Given  $a \in M$ , the grading (2.6) defines a family of anisotropic dilations  $\delta_t : x \rightarrow t \cdot x$ ,  $t > 0$ , on  $\mathfrak{g}M(a)$  given by

$$(2.16) \quad t \cdot X = t^w X \quad \forall X \in \mathfrak{g}_w M(a).$$

We note that the fact that  $\mathfrak{g}M(a)$  is a graded Lie algebra implies that

$$(2.17) \quad [t \cdot X, t \cdot Y](a) = t \cdot [X, Y] \quad \forall X, Y \in \mathfrak{g}M(a) \quad \forall t > 0.$$

The action of  $\delta_t$  on  $\text{Der}(\mathfrak{g}M(a))$  is given by

$$(2.18) \quad \delta_t(A) := (\delta_t)_* A = \delta_t \circ A \circ \delta_t^{-1}.$$

In particular, it follows from (2.13) and (2.17) that

$$\delta_t(\text{ad } X) = \text{ad}_{t \cdot X} \quad \forall X \in \mathfrak{g}M(a).$$



**Proposition 2.26.** *Let  $a \in M$  and  $t > 0$ . Then*

$$t \cdot (X \cdot Y) = (t \cdot X) \cdot (t \cdot Y) \quad \forall X, Y \in GM(a).$$

*Proof.* We note that if  $A$  and  $B$  are in  $\text{Der}(\mathfrak{g}M(a))$ , then  $\delta_t(AB) = \delta_t(A)\delta_t(B)$ . More generally, for any 2-variable power series  $g(z, y) = \sum a_{kl}z^k y^l$  we have  $\delta_t(g(A, B)) = g(\delta_t(A), \delta_t(B))$ . Applying this to  $g(z, y) = \int_0^1 \Phi(e^z e^{sy} - 1)ds$  and using (2.18) we get

$$\delta_t \left( \int_0^1 \Phi(e^{\text{ad}_X} e^{s \text{ad}_Y} - I) ds \right) = \int_0^1 \Phi(e^{\text{ad}_{t \cdot X}} e^{s \text{ad}_{t \cdot Y}} - I) ds.$$

Therefore, we see that  $t \cdot (X \cdot Y) = \delta_t(X \cdot Y)$  is equal to

$$\begin{aligned} \delta_t(X) + \delta_t \left( \int_0^1 \Phi(e^{\text{ad}_X} e^{s \text{ad}_Y} - I) ds \right) \delta_t(Y) &= t \cdot X + \left( \int_0^1 \Phi(e^{\text{ad}_{t \cdot X}} e^{s \text{ad}_{t \cdot Y}} - I) ds \right) (t \cdot Y) \\ &= (t \cdot X) \cdot (t \cdot Y). \end{aligned}$$

This proves the result.  $\square$

**2.4. Extrinsic description of  $GM(a)$ .** Let  $(X_1, \dots, X_n)$  be an  $H$ -frame near a point  $a \in M$ . As in (2.8) near  $a$  there are unique smooth functions  $L_{ij}^k(x)$ ,  $w_k \leq w_i + w_j$ , such that

$$[X_i, X_j] = \sum_{w_k \leq w_i + w_j} L_{ij}^k(x) X_k.$$

For  $i = 1, \dots, n$ , let  $\dot{X}_i(a)$  the class of  $X_i(a)$  in  $\mathfrak{g}_{w_i}M(a)$ . Then  $(\dot{X}_1(a), \dots, \dot{X}_n(a))$  is a basis of  $\mathfrak{g}M(a)$ , and hence defines coordinates  $(x_1, \dots, x_n)$  on  $\mathfrak{g}M(a)$ . In these coordinates the dilations (2.16) are given by

$$\delta_t(x_1, \dots, x_n) = (t^{w_1}x_1, \dots, t^{w_n}x_n), \quad t > 0.$$

Let  $X = \sum_{i \leq n} x_i \dot{X}_i(a)$  and  $Y = \sum_{i \leq n} y_i \dot{X}_i(a)$  be in  $\mathfrak{g}M(a)$ . Then using Remark 2.22 we get

$$\text{ad}_X Y = \sum_{i,j=1}^n x_i y_j [\dot{X}_i(a), \dot{X}_j(a)] = \sum_{i,j=1}^n \sum_{w_k = w_i + w_j} x_i y_j L_{ij}^k(a) \dot{X}_k(a).$$

This shows that the matrix  $A_a(x) = (A_a(x)_{kj})_{1 \leq j, k \leq n}$  of  $\text{ad}_X$  is given by

$$A_a(x)_{jk} = \begin{cases} \sum_{w_i = w_k - w_j} L_{ik}^j(a) x_i & \text{if } w_j > w_k, \\ 0 & \text{otherwise.} \end{cases}$$

Note that  $A_a(x)$  is a lower-triangular matrix. It then follows that in the coordinates  $(x_1, \dots, x_n)$  the product of  $GM(a)$  is given by

$$\begin{aligned} (2.19) \quad x \cdot y &= x + \left( \int_0^1 \Phi \left( e^{A_a(x)} e^{s A_a(y)} - I \right) ds \right) y \\ &= x + y + \frac{1}{2} A_a(x) y + \frac{1}{12} A_a(x)^2 y - \frac{1}{12} A_a(y) A_a(x) y + \dots \end{aligned}$$

When  $w_j = 1$  Eq. (2.19) gives

$$(x \cdot y)_j = x_j + y_j.$$

When  $w_j = 2$  we obtain

$$(x \cdot y)_j = x_j + y_j + \frac{1}{2} \sum_l A_a(x)_{jl} y_l = x_j + y_j + \frac{1}{2} \sum_{w_k = w_l = 1} L_{kl}^j(a) x_k y_l.$$

Likewise, when  $w_j = 3$  we get

$$\begin{aligned}(x \cdot y)_j &= x_j + y_j + \frac{1}{2} \sum_l A_a(x)_{jl} y_l + \frac{1}{2} \sum_{p,q} (A_a(x)_{jk} - A_a(y)_{jk}) A_a(x)_{kq} y_q \\ &= x_j + y_j + \frac{1}{2} \sum_{w_k+w_l=3} L_{kl}^j(a) x_k y_l + \frac{1}{12} \sum_{w_k=w_l=1} \sum_{w_p=w_q=1} L_{lk}^j(a) L_{pq}^k(a) (x_l - y_l) x_p y_q.\end{aligned}$$

More generally, we have

$$(2.20) \quad (x \cdot y)_j = x_j + y_j + \sum_{\substack{\langle \alpha \rangle + \langle \beta \rangle = w_j \\ \alpha \neq 0, \beta \neq 0}} b_{j\alpha\beta} x^\alpha y^\beta,$$

where  $b_{j\alpha\beta}$  is a universal polynomial in the structure constants  $L_{kl}^p(a)$ . Note also that the equality  $x^{-1} = x$  implies that  $x \cdot (-x) = 0$ , i.e.,

$$(2.21) \quad \sum_{\substack{\langle \alpha \rangle + \langle \beta \rangle = w_j \\ \alpha \neq 0, \beta \neq 0}} (-1)^{|\beta|} b_{j\alpha\beta} x^{\alpha+\beta} = 0.$$

It then follows that, for all  $j = 1, \dots, n$ , we have

$$\sum_{\alpha+\beta=\gamma} (-1)^{|\beta|} b_{j\alpha\beta} = 0 \quad \text{for all } \gamma \in \mathbb{N}_0^n, \langle \gamma \rangle = w_j.$$

Finally, we can interpret  $\mathfrak{g}M(a)$  as a Lie algebra of left-invariant vector fields on  $GM(a)$  as follows. For  $j = 1, \dots, n$ , let  $X_j^a$  be the left-invariant vector field on  $GM(a)$  that agrees at  $x = 0$  with  $\dot{X}_j(a)$  under the identification of  $\mathfrak{g}M(a)$  with the tangent space of  $GM(a)$  at 0. That is,

$$(2.22) \quad X_j^a f(x) = \left. \frac{d}{dt} f(x \cdot (t\dot{X}_j(a))) \right|_{t=0} \quad \forall f \in C^\infty(GM(a)).$$

The span of the vector fields  $X_j^a$  is a Lie algebra with same constant structures  $L_{ij}^k(a)$  as  $\mathfrak{g}M(a)$ . More precisely, as the Lie bracket  $[X_i^a, X_j^a]$  is the left-invariant vector field on  $GM(a)$  that agrees at  $x = 0$  with  $[\dot{X}_i(a), \dot{X}_j(a)](a) = \sum L_{ij}^k(a) \dot{X}_k(a)$ , we have

$$[X_i^a, X_j^a] = \sum_{w_k=w_i+w_j} L_{ij}^k(a) X_k^a.$$

In the coordinates  $(x_1, \dots, x_n)$  the vector fields  $X_j^a$  are computed as follows. For  $j = 1, \dots, n$ , set  $\epsilon_j = (\delta_{jk})_{1 \leq k \leq n} \in \mathbb{N}_0^n \subset \mathbb{R}^n$ . Then, for all  $f \in C^\infty(\mathbb{R}^n)$ , we have

$$X_j^a f(x) = \left. \frac{d}{dt} f(x \cdot (t\epsilon_j)) \right|_{t=0} = \sum_{1 \leq k \leq n} \left. \frac{d}{dt} (x \cdot (t\epsilon_j))_k \right|_{t=0} \partial_k f(x).$$

Using (2.20) we see that the  $k$ -th coordinate of  $x \cdot (t\epsilon_j)$  is given by

$$(x \cdot (t\epsilon_j))_k = x_k + t\delta_{jk} + \sum_{\substack{\langle \alpha \rangle + w_j = w_k \\ \alpha \neq 0}} t b_{j\alpha\epsilon_j} x^\alpha.$$

Combining this with (2.22) we then obtain

$$(2.23) \quad X_j^a = \partial_j + \sum_{\substack{w_k - \langle \alpha \rangle = w_j \\ \alpha \neq 0}} b_{jk\alpha} x^\alpha \partial_k \quad \text{where } b_{jk\alpha} := b_{k\alpha\epsilon_j}.$$

In particular, we see that the vector field  $X_j^a$  is homogeneous of degree  $-w_j$  with respect to the dilations  $\delta_t$ . That is,

$$\delta_t^* X_j^a = t^{-w_j} X_j^a \quad \text{for all } t > 0.$$

**2.5. The tangent map of a Carnot manifold map.** Let  $(M', H')$  be a step  $r$  Carnot manifold, so that  $H' = (H'_1, \dots, H'_r)$ , and let  $\phi : M \rightarrow M'$  be a Carnot manifold map. The condition (2.4) implies that, for  $w = 1, \dots, r$ , the tangent linear map  $\phi'$  induces a smooth vector bundle map,

$$\phi'_{[w]} : \mathfrak{g}_w M \longrightarrow \mathfrak{g}_{w'} M'.$$

More precisely, given  $a \in M$  and  $X \in H_w(a)$ , if we denote by  $\dot{X}(a)$  the class of  $X(a)$  modulo  $H_{w-1}(a)$ , then  $\phi'_{[w]}(a) \left( \dot{X}(a) \right)$  is equal to the class of  $\phi'(a)(X(a))$ . The value of this class does change when  $X(a)$  remains in the class modulo  $H_{w-1}(a)$ .

Gathering together the maps  $\phi'_{[w]}$ ,  $w = 1, \dots, r$ , we obtain a smooth vector bundle map,

$$\phi'_H : \mathfrak{g}M \longrightarrow \mathfrak{g}M', \quad \phi'_H = \phi'_{[1]} \oplus \dots \oplus \phi'_{[r]}.$$

**Proposition 2.27.** *The vector bundle map  $\phi'_H$  is a smooth Carnot-type algebra bundle map from  $\mathfrak{g}M$  to  $\mathfrak{g}M'$ .*

*Proof.* It is immediate from its construction that  $\phi'_H$  is compatible with the grading (2.6), i.e.,  $\phi'_H(\mathfrak{g}_w M) \subset \mathfrak{g}_{w'} M'$  for  $w = 1, \dots, r$ . Therefore, we only have to check the compatibility with the Lie bracket. Let  $a \in M$  and set  $a' = \phi(a)$ . In addition, let  $X$  (resp.,  $Y$ ) be a smooth section of  $H_w$  (resp.,  $H_{w'}$ ) on a neighborhood of  $a$  with  $w + w' \leq r$ . As above we shall use a dot to denote classes in  $\mathfrak{g}M$  or  $\mathfrak{g}M'$ . By definition of  $\phi'_H$  we have

$$\phi'_H(a) \left( \dot{X}(a) \right) = \phi'_{[w]} \left( \dot{X}(a) \right) = \phi'(a)(X(a)) = \phi_* X(a') \mod \mathfrak{g}_{w'} M'(a').$$

Likewise, we have  $\phi'_H(a) \left( \dot{Y}(a) \right) = \phi_* Y(a') \mod \mathfrak{g}_{w'} M'(a')$ . Thus,

$$\begin{aligned} \left[ \phi'_H(a) \left( \dot{X}(a) \right), \phi'_H(a) \left( \dot{Y}(a) \right) \right] (a') &= \mathcal{L}_{w, w'}(a') \left( \phi'_H(a) \left( \dot{X}(a) \right), \phi'_H(a) \left( \dot{Y}(a) \right) \right) \\ &= [\phi_* X, \phi_* Y](a') \mod \mathfrak{g}_{w+w'} M'(a'). \end{aligned}$$

We also note that by definition  $[\dot{X}(a), \dot{Y}(a)](a) = \mathcal{L}_{w, w'}(a) \left( \dot{X}(a), \dot{Y}(a) \right)$  is the class of  $[X, Y](a)$  modulo  $H_{w+w'-1}(a)$ . Therefore, we have

$$\phi'_H(a) \left( [\dot{X}(a), \dot{Y}(a)](a) \right) = \phi'(a) ([X, Y](a)) = \phi_* [X, Y](a') \mod \mathfrak{g}_{w+w'} M'(a').$$

As  $\phi_* [X, Y] = [\phi_* X, \phi_* Y]$  we then deduce that

$$\phi'_H(a) \left( [\dot{X}(a), \dot{Y}(a)](a) \right) = \left[ \phi'_H(a) \left( \dot{X}(a) \right), \phi'_H(a) \left( \dot{Y}(a) \right) \right] (a').$$

This proves the compatibility of  $\phi'_H$  with the Lie bracket. The proof is complete.  $\square$

*Remark 2.28.* The compatibility of  $\phi'_H$  with the Carnot grading (2.6) implies that

$$\phi'_H(a)(t \cdot X) = t \cdot \phi'_H(a)(X) \quad \text{for all } (a, X) \in GM \text{ and } t > 0.$$

As the exponential maps  $\mathfrak{g}M \rightarrow GM$  and  $\mathfrak{g}M' \rightarrow GM'$  are defined to be identity maps, Proposition 2.27 immediately implies the following statement.

**Proposition 2.29.** *If we regard  $\phi'_H$  as a map from  $GM$  to  $GM'$ , then we obtain a smooth Carnot-type group bundle map.*

We also mention the following result.

**Proposition 2.30.** *Let  $(M'', H'')$  be a step  $r$  Carnot manifold and  $\psi : M' \rightarrow M''$  a Carnot manifold map. Then we have*

$$(\psi \circ \phi)'_H = \psi'_H \circ \phi'_H.$$

*Proof.* Let  $a \in M$  and set  $a' = \phi(a)$  and  $a'' = \psi \circ \phi(a)$ . In addition, let  $w \in \{1, \dots, r\}$  and  $X$  a local section of  $H_w$  near  $a$ . By definition  $\phi'_H \left( \dot{X}(a) \right)$  is the class of  $\phi'(a)(X(a)) = \phi_* X(a')$  modulo  $H'_{w-1}(a')$ . Therefore, we get

$$\psi'_H(a') \left( \phi'_H(a) \left( \dot{X}(a) \right) \right) = \psi'(a'') (\phi_* X(a')) = (\psi \circ \phi)_* X(a'') \mod H''_{w-1}(a'').$$

We also know that  $(\psi \circ \phi)'_H(a) \left( \dot{X}(a) \right)$  is the class of  $(\psi \circ \phi)_* X(a'') = (\psi \circ \phi)_*(a'')$  modulo  $H''_{w-1}(a'')$ . Therefore, we deduce that

$$(\psi \circ \phi)'_H(a) \left( \dot{X}(a) \right) = \psi'_H(a') \circ \phi'_H(a) \left( \dot{X}(a) \right).$$

This shows that  $(\psi \circ \phi)'_H = \psi'_H \circ \phi'_H$ . The proof is complete.  $\square$

The above results show that the constructions of the tangent Lie algebra bundle and tangent group bundle are functorial. In particular, in the case of Carnot diffeomorphisms we obtain the following result.

**Proposition 2.31.** *Assume that  $(M', H')$  has same type as  $(M, H)$  and let  $\phi : M \rightarrow M'$  be a Carnot diffeomorphism. Then*

- (1)  $\phi'_H : \mathfrak{g}M \rightarrow \mathfrak{g}M'$  is an isomorphism of Carnot-type algebra bundles.
- (2)  $\phi'_H : GM \rightarrow GM'$  is an isomorphism of Carnot-type group bundles.
- (3) The inverse map of  $\phi'_H$  is given by

$$\phi'_H(a)^{-1} = (\phi^{-1})'_H(\phi(a)) \quad \text{for all } a \in M.$$

### 3. EXAMPLES OF CARNOT MANIFOLDS

In this section, we describe various examples of Carnot manifolds. The following list is by no means exhaustive, but it should give to the reader a glimpse at the vast diversity of examples of Carnot manifolds.

**3.1. Nilpotent Lie Groups.** Let  $G$  be a step  $r$  Carnot-type group. Recall that by Definition 2.3 this means that  $G$  is a simply connected nilpotent Lie group whose Lie algebra  $\mathfrak{g}$  has a grading,

$$(3.1) \quad \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_r,$$

in such a way that

$$(3.2) \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \text{ for } i+j \leq r \quad \text{and} \quad [\mathfrak{g}_i, \mathfrak{g}_j] = \{0\} \text{ for } i+j > r.$$

The grading (3.1) gives rise to a family of anisotropic dilations  $\delta_t$ ,  $t > 0$ , given by

$$\delta_t(\xi_1 + \xi_2 + \cdots + \xi_r) = t\xi_1 + t^2\xi_2 + \cdots + t^r\xi_r, \quad \xi_j \in \mathfrak{g}_j.$$

Thanks to (3.2) these dilations are Lie algebra automorphisms of  $\mathfrak{g}$ , and so they give rise to Lie group isomorphisms of  $G$ . For  $j = 1, \dots, r$ , set  $\mathfrak{h}_j = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_j$ . Using the identification  $\mathfrak{g} \simeq T_0G$ , let  $H_j$  be the left-invariant subbundle of  $TG$  that agrees with  $\mathfrak{h}_j$  at  $x = 0$ . Then  $H_r = TG$  and  $[H_i, H_j] \subset H_{i+j}$  for  $i+j \leq r$ . Therefore, we get a left-invariant Carnot flag  $H = (H_0, \dots, H_r)$  over  $G$  (where we have set  $H_0 = \{0\}$ ).

The simplest non-Abelian Carnot group is the Heisenberg group  $\mathbb{H}^{2n+1}$ . It is realized as  $\mathbb{R}^{2n+1}$  equipped with the group law,

$$(3.3) \quad x \cdot y = \left( x_0 + y_0 + \sum_{1 \leq j \leq n} (x_j y_{n+j} - x_{n+j} y_j), x_1 + y_1, \dots, x_{2n} + y_{2n} \right).$$

A tangent frame is provided by the left-invariant vector fields,

$$X_0 = \partial_{x_0}, \quad X_j = \partial_{x_j} - x_{n+j} \partial_{x_0}, \quad X_j = \partial_{x_{n+j}} + x_{n+j} \partial_{x_0}, \quad j = 1, \dots, n.$$

We note that, for  $j, k = 1, \dots, n$ , we have

$$\begin{aligned} [X_j, X_0] &= [X_{n+j}, X_0] = [X_j, X_k] = [X_{n+j}, X_{n+k}] = 0, \\ [X_j, X_k] &= 2\delta_{n+j,k} X_0 \quad \text{and} \quad [X_j, X_{n+k}] = 0 \quad \text{if } k \neq n+j. \end{aligned}$$

A left-invariant Carnot flag then is  $(H, TH^{2n+1})$ , where  $H$  is the hyperplane bundle generated by  $X_1, \dots, X_{2n}$ . In addition, the dilations  $\delta_t$  are simply given by

$$(3.4) \quad \delta_t(x) = (t^2 x_0, t x_1, \dots, t x_{2n}), \quad t > 0.$$

Another example of Carnot group is the Engel group  $\mathcal{E}$ . This is the 3-step nilpotent group obtained by equipping  $\mathbb{R}^4$  with the group law,

$$x \cdot y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1 y_2, x_4 + y_4 + x_3 y_2).$$

A left-invariant tangent frame is given by

$$X_1 = \partial_{x_1}, \quad X_2 = \partial_{x_2} + x_1 \partial_{x_3} + x_3 \partial_{x_4}, \quad X_3 = \partial_{x_3}, \quad X_4 = \partial_{x_4}.$$

We note that  $[X_1, X_2] = X_3$  and  $[X_2, X_3] = [X_2, [X_1, X_2]] = X_4$ . A Carnot flag then is  $(H_1, H_2, H_3)$ , where  $H_j$ ,  $j = 1, 2, 3$ , is the subbundle generated by  $X_1, \dots, X_{j+1}$ . This defines a left-invariant Carnot structure on  $\mathcal{E}$  of type  $(2, 3, 4)$ .

**3.2. Heisenberg Manifolds.** In the terminology of [BG], a Heisenberg manifold is a manifold together with a distinguished hyperplane bundle  $H \subset TM$ . In particular,  $(H, TM)$  is a Carnot flag. Following are important examples of Heisenberg manifolds:

- The Heisenberg group  $\mathbb{H}^{2n+1}$  and its products with Abelian groups.
- Cauchy-Riemann (CR) manifolds of hypersurface type, e.g., real hypersurfaces in complex manifolds (see Subsection 3.5 below).
- Contact manifolds and even contact manifolds.
- Confoliations of Eliashberg-Thurston [ET].

Given a Heisenberg manifold  $(M, H)$ , taking Lie bracket of horizontal vector fields modulo  $H$  defines an antisymmetric bilinear vector bundle map  $\mathcal{L} : H \times H \rightarrow TM/H$ , which is called the Levi form of  $(M, H)$  (see, e.g., [Po1]). If  $M$  has odd dimension, then  $H$  is a contact distribution if and only if the Levi form is non-degenerate at every point. If  $\dim M$  is even, we say that  $(M, H)$  is an even contact manifold when the Levi form has maximal rank  $\dim M - 2$  at every point (cf. [Mo]).

**3.3. Foliations and Polycontact Manifolds.** Generalizing the example of a Heisenberg manifold we can consider a manifold  $M$  equipped with a Carnot flag  $(H, TM)$  where  $H$  is a subbundle of  $TM$  of arbitrary corank. In this case too there is a well defined Levi form  $\mathcal{L} : H \times H \rightarrow TM/H$ . There are two important opposite examples of such structures:

- Foliations, in which case  $H$  is integrable in Fröbenius' sense, i.e.,  $[H, H] \subset H$ , or equivalently, the Levi form  $\mathcal{L}$  vanishes at every point. An important example of foliation is the Kröner foliation on the 2-torus  $\mathbb{T}^2$ .
- Polycontact manifolds, in which case  $H$  is totally non-integrable at every point. That is, for all  $x \in M$  and  $\theta \in (T_x M / H_x)^* \setminus 0$ , the bilinear form  $\theta \circ \mathcal{L}_x : H_x \times H_x \rightarrow \mathbb{R}$  is non-degenerate. Following are important examples of polycontact manifolds:
  - Métivier groups [Mé], including the  $H$ -type groups of Kaplan [Ka].
  - Principal bundles equipped with the horizontal distribution defined by a fat connection (cf. Weinstein [We]).
  - The quaternionic contact manifolds of Biquard [Bi1, Bi2].
  - The unit sphere  $\mathbb{S}^{4n-1}$  in quaternionic space (see [vE3]).

*Remark 3.1.* Polycontact manifolds are called Heisenberg manifolds in [CC]. Polycontact distributions are called fat in [Mo]. The terminology polycontact was coined by van Erp [vE3] (following a suggestion of Alan Weinstein).

**3.4. Carnot-Carathéodory Manifolds.** In what follows, given a manifold  $M$  and distributions  $H_j \subset TM$ ,  $j = 1, 2$ , we shall denote by  $[H_1, H_2]$  the distribution generated by the Lie brackets of their sections, i.e.,

$$[H_1, H_2] = \bigsqcup_{x \in M} \{[X_1, X_2](x); X_j \in C^\infty(M, H_j), j = 1, 2\}.$$

In particular, given a subbundle (or even distribution)  $H \subset TM$ , we recursively define distributions  $H^{[j]}$ ,  $j \geq 1$ , by

$$H^{[1]} = H \quad \text{and} \quad H^{[j+1]} = H^{[j]} + [H, H^{[j]}], \quad j \geq 1.$$

**Definition 3.2.** A *Carnot-Carathéodory* manifold is a pair  $(M, H)$ , where  $M$  is a manifold and  $H$  is a subbundle of  $TM$  such that, for all  $x \in M$ , we can find  $r \in \mathbb{N}$  so that  $H^{[r]}(a) = TM(x)$ .

**Definition 3.3** ([Gro]). A Carnot-Carathéodory manifold  $(M, H)$  is called *equiregular* when there is  $r \in \mathbb{N}$  such that  $H^{[r]} = TM$  and each distribution  $H^{[j]}$ ,  $j = 2, \dots, r-1$ , has constant rank.

Any equiregular Carnot-Carathéodory manifold  $(M, H)$  is a Carnot manifold with Carnot flag  $(H^{[1]}, \dots, H^{[r]})$ . Moreover, any non-equiregular Carnot-Carathéodory structure can be "desingularized" into an equiregular Carnot-Carathéodory structure (see, e.g., [Je2]).

Contact and polycontact manifolds are examples of equiregular Carnot-Carathéodory manifolds with  $r = 2$ . An important example of equiregular Carnot-Carathéodory manifolds with  $r \geq 3$  is provided by Engel manifolds.

**Definition 3.4.** An Engel manifold is a 4-dimensional manifold equipped with a plane-bundle  $H \subset TM$  such that  $H^{[2]}$  has constant rank 3 and  $H^{[3]} = TM$ .

The Engel group described above is an Engel manifold and provides us with the local model of an Engel manifold. Other examples of Engel manifolds occur from prolongations of contact manifolds (see [Mo]). As shown by Vogel [Vo] any parallelizable 4-manifold can be endowed with an Engel manifold structure.

Carnot-Carathéodory structures appear in various real-life situations such as skating motion [Be, Bl], rolling penny [CC], car-like robots [CC], or car with  $N$ -trailers [Je1]. They also appear in numerous applied mathematics settings such as the Asian option model in finance, computer vision, image processing, dispersive groundwater and pollution, statistical properties of laser light, extinction in systems of interacting biological populations, dynamics of polymers and astronomy (distribution of clusters in space) (see [Br] and the references therein).

We refer to [ABB, CC, Gro, Je2, Mo, Ri], and the references therein, for detailed accounts on Carnot-Carathéodory structures and their numerous applications.

**3.5. Cauchy-Riemann Manifolds.** Cauchy-Riemann manifolds (a.k.a. CR manifolds) play an important role in several complex variables. A CR structure on a manifold  $M$  is given by the datum of a subbundle  $H \subset TM$  which carries an integrable complex structure. For instance, let  $J_0$  the canonical complex structure of  $\mathbb{C}^N$  induced by the multiplication by  $i$ . Then  $J_0$  induces a CR structure on a given real submanifold  $M \subset \mathbb{C}^N$  if and only if the distribution  $H := TM \cap J_0(TM)$  has constant rank (i.e.,  $H$  is a subbundle). In particular, any real hypersurface  $H \subset \mathbb{C}^N$  is a CR submanifold, since in this case  $H$  is always a hyperplane bundle. More generally, any abstract CR manifold  $(M, H)$  where  $H$  is hyperplane bundle is said to be of hypersurface type.

A CR manifold  $(M, H)$  has finite type  $r$  at a given point  $x_0$  when  $H^{[r]}(x_0) = TM(x_0)$  (see [Ko, BGr]). Therefore, it has finite type  $r$  everywhere if and only if  $H^{[r]} = TM$ , in which case  $H$  defines a step  $r$  Carnot-Carathéodory structure. A real hypersurface of  $\mathbb{C}^N$ , and more generally any hypersurface type CR manifold, has finite type 2 everywhere if and only if the associated Levi form is nowhere zero. Moreover, a real analytic CR manifold  $M \subset \mathbb{C}^N$  has finite type at a point  $x_0 \in M$  if and only if it is minimal at  $x_0$ , in the sense that it does not exist a submanifold  $S \subset N$  containing  $x_0$  such that  $H(x_0) \subset TS(x_0)$  and  $\dim S < \dim M$  (see [BER, Thm. 4.1.3]).

**3.6. Parabolic geometric structures.** A number of important geometric examples of Carnot-Carathéodory structures arise from parabolic structures. We shall briefly explain how such structures occur. For a thorough account on parabolic geometry we refer to the monograph [CS] and the references therein. The terminology parabolic geometry was coined by Graham [Gr] in the context of extending the program of Fefferman [Fe] in CR geometry to various natural geometric settings.

Let  $G$  be a complex semisimple Lie group with Lie algebra  $\mathfrak{g}$ . Let  $\mathfrak{h}$  be a Cartan subalgebra of  $\mathfrak{g}$  and denote by  $\Delta$  its associated root system. Thus,  $\Delta$  consists of  $\alpha \in \mathfrak{h}^* \setminus \{0\}$  with non-zero root space  $g_\alpha = \{X \in \mathfrak{g}; [X, Y] = \alpha(Y)X \ \forall Y \in \mathfrak{h}\}$ . Recall that if  $\alpha$  is a root, then so is  $-\alpha$  and the associated root space is one-dimensional. Moreover, for all  $\alpha$  and  $\beta$  in  $\Delta$ , we have

$$[g_\alpha, g_\beta] = \begin{cases} g_{\alpha+\beta} & \text{if } \alpha + \beta \in \Delta, \\ \{0\} & \text{otherwise.} \end{cases}$$



In addition, we have the decomposition,

$$(3.5) \quad \mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha}.$$

In the following, we let  $\Delta^+$  be a system of positive roots and denote by  $\Delta^0 \subset \Delta^+$  the associated set of (positive) simple roots. Any root then is a linear combination with integer coefficients of simple roots in such way that all coefficients have same sign.

Let  $\mathfrak{b} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{\alpha}$  be the Borel subalgebra associated with  $\Delta^+$ . In what follows, we let  $\mathfrak{p}$  be a subalgebra containing  $\mathfrak{b}$  and denote by  $P$  its Lie group. Such a subalgebra is called a *parabolic* subalgebra. Parabolic subalgebras containing  $\mathfrak{b}$  are in one-to-one correspondence with subsets of  $\Delta^0$  (see [CS]). In particular, let  $\Sigma$  be the subset of  $\Delta^0$  consisting of simple roots such that  $\mathfrak{g}_{-\alpha}$  is not contained in  $\mathfrak{p}$ . The associated weight (or height) function  $w : \Delta^0 \rightarrow \mathbb{Z}$  is defined as follows. Let  $\{\alpha_1, \dots, \alpha_m\}$  be a listing of the elements of  $\Delta^0$ . Any positive root  $\alpha \in \Delta^+$  is uniquely written as  $\alpha = \sum_{j=1}^m \lambda_j \alpha_j$  with  $\lambda_j \in \mathbb{N}_0$ . The weight of  $\alpha$  is then defined by

$$w(\alpha) = \sum_{\alpha_j \in \Sigma} \lambda_j.$$

Let  $\Delta_{\mathfrak{p}}^+$  be the set of positive roots with zero weight, i.e., positive roots that are linear combinations of roots in  $\Delta^0 \setminus \Sigma$ . We then have

$$(3.6) \quad \mathfrak{p} = \mathfrak{b} \oplus \left( \bigoplus_{\alpha \in \Delta_{\mathfrak{p}}^+} \mathfrak{g}_{-\alpha} \right).$$

Combining (3.5) and (3.6) we see that  $\mathfrak{g}/\mathfrak{p}$  is naturally identified with the nilpotent subalgebra,

$$(3.7) \quad \mathfrak{g} = \mathfrak{p} \oplus \mathfrak{n}, \quad \text{where } \mathfrak{n} = \bigoplus_{\alpha \in \Delta^0 \setminus \Delta_{\mathfrak{p}}^+} \mathfrak{g}_{-\alpha} = \bigoplus_{\substack{\alpha \in \Delta^0 \\ w(\alpha) \neq 0}} \mathfrak{g}_{-\alpha}.$$

The weight function  $w$  provides  $\mathfrak{n}$  with a natural grading,

$$\mathfrak{n} = \bigoplus_{1 \leq j \leq r} \mathfrak{n}_j, \quad \text{where } \mathfrak{n}_j := \bigoplus_{\substack{\alpha \in \Delta^0 \\ w(\alpha) = j}} \mathfrak{g}_{-\alpha}.$$

Here  $r$  is the maximal weight of a root. We also note that, for  $i, j = 1, \dots, r$ , we have

$$[\mathfrak{n}_i, \mathfrak{n}_j] = \mathfrak{n}_{i+j} \quad \text{if } i + j \leq r.$$

In particular, we see that  $\mathfrak{n}$  is generated by  $\mathfrak{n}_1$ , and so when  $r \geq 2$  we see that  $\mathfrak{n}$  is a Carnot algebra in the sense of Pansu [Pa].

The Levi subgroup  $G_0 \subset P$  is the Lie group associated with the Levi subalgebra,

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_{\mathfrak{p}}^+} \mathfrak{g}_{-\alpha}.$$

We note that  $[\mathfrak{g}_0, \mathfrak{n}_j] \subset \mathfrak{n}_j$  for  $j = 1, \dots, r$ . Therefore, if we denote by  $\text{Aut}^{\natural}(\mathfrak{n})$  the group of Lie algebra automorphisms of  $\mathfrak{n}$  preserving the grading (3.7), then the adjoint action of  $G_0$  induces a group homomorphism,

$$(3.8) \quad \text{Ad} : G_0 \longrightarrow \text{Aut}^{\natural}(\mathfrak{n}).$$

In fact, this property characterizes  $G_0$  (see [CS]).

**Definition 3.5.** A parabolic structure of type  $\mathfrak{g}/\mathfrak{p}$  on a manifold  $M$  is given by

- (i) A Carnot flag  $H = (H_0, H_1, \dots, H_r)$  such that the associated tangent Lie algebra bundle  $\mathfrak{g}M$  is an  $\mathfrak{n}$ -principal Lie algebra bundle.
- (ii) A reduction to  $G_0$  of the structure group of  $\mathfrak{g}M$  (seen as an  $\mathfrak{n}$ -principal Lie algebra bundle).

*Remark 3.6.* The condition (i) simply means that, near any point  $a \in M$ , there is a local tangent frame  $\{X_{\alpha}\}_{\alpha \in \Delta^+ \setminus \Delta_{\mathfrak{p}}^+}$  such that

- $\{X_{\alpha}; w(\alpha) = j\}$  is a local frame of  $H_j$ .

- For all  $\alpha$  and  $\beta$  in  $\Delta^+ \setminus \Delta_{\mathfrak{p}}^+$ , we have

$$[X_\alpha, X_\beta] = \begin{cases} X_{\alpha+\beta} & \text{mod } H_{w(\alpha)+w(\beta)-1} \quad \text{if } \alpha + \beta \text{ is a root,} \\ 0 & \text{mod } H_{w(\alpha)+w(\beta)-1} \quad \text{otherwise.} \end{cases}$$

*Remark 3.7.* Assuming that  $\mathfrak{g}M$  is an  $\mathfrak{n}$ -principal Lie algebra bundle, its frame bundle is defined as the  $\text{Aut}^{\mathfrak{h}}(\mathfrak{n})$ -principal bundle  $\mathcal{F}^{\mathfrak{h}}(\mathfrak{g}M)$  over  $M$  whose fiber at a point  $x \in M$  consists of all graded Lie algebra automorphisms  $\mathfrak{n} \rightarrow \mathfrak{g}_x M$ . A reduction to  $G_0$  of the structure group of  $\mathfrak{g}M$  is then given by a  $G_0$ -principal bundle  $\mathcal{G}_0$  and an identification of  $\text{Aut}^{\mathfrak{h}}(\mathfrak{n})$ -principal bundles,

$$\mathcal{F}^{\mathfrak{h}}(\mathfrak{g}M) \simeq \text{Aut}^{\mathfrak{h}}(\mathfrak{n}) \times_{G_0} \mathcal{G}_0,$$

where  $\text{Aut}^{\mathfrak{h}}(\mathfrak{n}) \times_{G_0} \mathcal{G}_0$  is the bundle associated with  $\mathcal{G}_0$  and the homomorphism (3.8).

*Remark 3.8.* In various cases the homomorphism (3.8) is an isomorphism. In those cases a parabolic structure is simply given by a Carnot structure whose associated tangent Lie algebra bundle is an  $\mathfrak{n}$ -principal Lie algebra bundle.

*Remark 3.9.* In the case  $r = 1$ , the Lie algebra  $\mathfrak{n}$  is just a vector space and so  $\mathfrak{g}M$  agrees with the tangent space  $TM$ . Therefore, in this case a parabolic structure is simply given by a reduction of to  $G_0$  of the structure group of  $TM$ , i.e., a  $G_0$ -structure. Although such parabolic structures do not produce Carnot structures *stricto sensu*, it is worth mentioning that such geometric structures naturally appear in the context of conformal geometry, projective differential geometry and almost Hermitian geometry (see [CS]).

*Remark 3.10.* We refer to [CS] and the references therein for an interpretation of geometric parabolic structures in terms of Cartan connections.

When  $r \geq 2$  parabolic structures produce true Carnot structures. Examples of such structures include the following:

- Nondegenerate partially integrable CR manifolds of hypersurface type (see below).
- Contact path geometric structures, including Contact projective structures (*cf.* [Fo1, Fo2]).
- Contact quaternionic structures of Biquard [Bi1, Bi2].
- Generic  $(m, \frac{1}{2}m(m+1))$ -distributions, including generic  $(3, 6)$ -distributions studied by Bryant [Bry] (see [CS]).
- Generic  $(2, 3, 5)$ -distributions introduced by Cartan [Ca].

A non-degenerate partially integrable CR structure (of hypersurface type) on a oriented manifold  $M^{2n+1}$  is given by a contact distribution  $H \subset TM$  and an almost complex structure on  $H$  that preserves the Levi form  $\mathcal{L} : H \times H \rightarrow TM/H$ . Such structures are geometric parabolic structures of type  $\mathfrak{su}(p+1, q+1)/\mathfrak{p}$  where  $p+q = n$  and  $\mathfrak{p}$  is a maximal parabolic subalgebra (see [CS]).

**3.7. Non-Carathéodory Examples.** The example of a Carnot structure associated with a foliation is an instance of Carnot structure that is not a Carnot-Carathéodory structure. In fact, it should be stressed out that even for studying Carnot-Carathéodory structures we may be naturally lead to study non-Carathéodory Carnot structures.

A first instance of such a situation occurs when studying the heat operator  $\Delta + \partial_t$  associated with a sum of squares  $\Delta = -(X_1^2 + \dots + X_m^2)$ , where  $X_1, \dots, X_m$  are real vector fields on a manifold  $M$  satisfying the bracket condition of Hörmander [Hö]. Therefore, the distribution  $H$  generated by the vector fields  $X_1, \dots, X_m$  gives rise to a Carnot-Carathéodory structure  $H = (H^{[1]}, H^{[2]}, \dots, H^{[r]})$  on  $M$ . For sake of simplicity, we shall assume this structure to be equiregular. The manifold  $M \times \mathbb{R}$  then carries a natural Carnot structure defined as follows. Let  $\pi_1$  (resp.,  $\pi_2$ ) be the projection of  $M \times \mathbb{R}$  onto the first factor  $M$  (resp., second factor  $\mathbb{R}$ ), and define

$$\tilde{H}_1 = \pi_1^* H, \quad \tilde{H}_j = \pi_1^* H^{[j]} + \pi_2^* T\mathbb{R}, \quad j \geq 2.$$

Then the flag  $\tilde{H} = (\tilde{H}_1, \dots, \tilde{H}_r)$  defines a Carnot structure on  $M \times \mathbb{R}$ . This is not a Carnot-Carathéodory structure since  $(\tilde{H}_1)^{[j]} = \pi_1^* H^{[j]} \neq \tilde{H}_j$  for all  $j \geq 2$ .

Other instances of non-Carathéodory structure naturally occur when studying the action of an arbitrary group of diffeomorphisms preserving a given Carnot structure. More precisely, let  $(M, H)$  be a Carnot manifold and denote by  $G$  the associated group of Carnot-diffeomorphisms, i.e., the group of smooth diffeomorphisms of  $M$  preserving each subbundle  $H_j$  of the flag  $H$ . The quotient  $M/G$  need not be Hausdorff. A solution provided by noncommutative geometry [Co2, CM] to study the action of  $G$  is to replace by the badly behaved space  $M/G$  by a “spectral triple”  $(\mathcal{A}, \mathcal{H}, D)$ . Without getting into too much details, the construction of such a spectral triple usually requires the passage to the total space of a  $G$ -invariant fibration  $P \rightarrow M$ .

For instance, when  $G$  is the full group of diffeomorphisms and  $H$  is the trivial step 1 flag  $(TM)$ , there is no  $G$ -invariant metric on  $M$  (in fact the differentiable structure of  $M$  is the only geometric structure preserved by  $G$ !). Let  $P \xrightarrow{\pi} M$  be the ray-bundle of metrics of  $M$  whose fibers consist of positive-definite symmetric  $(0, 2)$ -tensors  $g_{ij}dx^i \otimes dx^j$ . As observed by Connes [Co1], the total space of  $P$  carries a wealth of  $G$ -invariant objects. In particular, the vertical bundle  $V := \ker d\pi$  gives rise to a  $G$ -invariant (integrable) Carnot structure  $(V, TP)$  on the total manifold  $P$  and the bundle  $V \oplus \pi^*TM \simeq TP$  carries a  $G$ -invariant metric with respect to which  $V$  and  $\pi^*TM$  are orthogonal. Furthermore, the construction of the corresponding spectral triple deeply relies on the hypoelliptic pseudodifferential calculus associated with the pair  $(P, V)$  (see [CM]).

The above construction can be extended to more general Carnot manifolds. For sake of simplicity, we assume that  $(M, H)$  is an orientable contact manifold and  $G$  is its contactomorphism group. As the group  $G$  is essential (see, e.g., [Ba]), there is no  $G$ -invariant metric on  $M$ . The contact analogue of the metric bundle is the bundle of contact metrics, i.e., metrics of the form,

$$g = g_H \oplus \theta \otimes \theta,$$

where  $g_H$  is a positive-definite metric on  $H$  and  $\theta$  is a contact form (i.e., a non-zero section of  $(TM/H)^*$ ). Fixing a contact form  $\theta$  on  $M$ , this bundle can be realized as

$$P = P(H) \oplus \mathbb{R}_+^* \theta \xrightarrow{\pi} M,$$

where  $P(H)$  is the metric bundle of  $H$  and  $\mathbb{R}_+^* \theta$  is the ray bundle spanned by the contact form  $\theta$ . Let  $V = \ker d\pi \subset TP$  be the vertical bundle of the fibration  $\pi : P \rightarrow M$ . Note that  $V$  is contained in hyperplane bundle  $\tilde{H} = \ker \pi^* \theta \subset TP$ , where  $\theta$  is the pullback of  $\theta$  to  $P$ . Then the  $G$ -invariant flag  $(V, \tilde{H}, \tilde{H}, TM)$  naturally defines a Carnot structure on the total manifold  $P$ . This is not a Carnot-Carathéodory structure, since the subbundle  $V$  is integrable. In general, even if we start with a(n equiregular) Carnot-Carathéodory manifold  $(M, H)$ , then the Carnot structure on the “Carnot metric bundle”  $P$  is never a Carnot-Carathéodory structure.

#### 4. PRIVILEGED COORDINATES FOR CARNOT MANIFOLDS

In this section, we explain how to adapt to the setting of Carnot manifolds the construction of privileged coordinates of Bellaïche [Be] for Carnot-Carathéodory manifolds. We refer to [AS, Be, Go, He, RS], and the references therein, for various types of construction of privileged coordinates.

In what follows we let  $(X_1, \dots, X_n)$  be an  $H$ -frame on an open neighborhood  $V$  of a given point  $a \in M$ . Then there are unique smooth functions  $L_{ij}^k(x)$  on  $V$  such that

$$(4.1) \quad [X_i, X_j] = \sum_{w_k \leq w_i + w_j} L_{ij}^k(x) X_k.$$

In addition, given any finite sequence  $I = (i_1, \dots, i_k)$  with values in  $\{1, \dots, n\}$ , we define

$$X_I = X_{i_1} \cdots X_{i_k}.$$

For such a sequence we also set  $|I| = k$  and  $\langle I \rangle = w_{i_1} + \cdots + w_{i_k}$ .

**Definition 4.1.** Let  $f(x)$  be a smooth function defined near  $x = a$  and  $N$  a non-negative integer.

- (1) We say that  $f(x)$  has order  $\geq N$  at  $a$  when  $X_I f(a) = 0$  whenever  $\langle I \rangle < N$ .
- (2) We say that  $f(x)$  has order  $N$  at  $a$  when it has order  $\geq N$  and there is a sequence  $I = (i_1, \dots, i_k)$  with values in  $\{1, \dots, n\}$  with  $\langle I \rangle = N$  such that  $X_I f(a) \neq 0$ .

*Remark 4.2.* The above definition of the order of a function differs from that of Bellaïche [Be] as Bellaïche only considers monomials in vector fields  $X_i$  with  $w_i = 1$ .

**Lemma 4.3.** *Let  $f(x)$  be a smooth function near  $x = a$ . Then its order is independent of the choice of the  $H$ -frame  $(X_1, \dots, X_n)$  near  $a$ .*

*Proof.* Let  $(Y_1, \dots, Y_n)$  be another  $H$ -frame near  $a$ . We note that each vector field  $Y_i$  is a section of  $H_i$ . Therefore, near  $x = a$ , we may write

$$Y_i = \sum_{w_j \leq w_i} c_{ij}(x) X_j,$$

where the coefficients  $c_{ij}(x)$  are smooth and there is an integer  $j$  with  $w_j = w_i$  in such that  $c_{ij}(a) \neq 0$ . More generally, given any finite sequence  $I = (i_1, \dots, i_k)$  with values in  $\{1, \dots, n\}$ , near  $x = a$  we further may write

$$(4.2) \quad Y_I = Y_{i_1} \cdots Y_{i_k} = \left( \sum_{w_{j_1} \leq w_{i_1}} c_{i_1 j_1}(x) X_{j_1} \right) \cdots \left( \sum_{w_{j_k} \leq w_{i_k}} c_{i_k j_k}(x) X_{j_k} \right) = \sum_{\langle J \rangle \leq \langle I \rangle} c_{IJ}(x) X_J,$$

for some smooth coefficients  $c_{IJ}(x)$ .

Let  $N$  be the order of  $f$  with respect to the  $H$ -frame  $(X_1, \dots, X_n)$ . If  $\langle I \rangle < N$ , then (4.2) shows that  $Y_I f(a)$  is a linear combination of terms  $X_J f(a)$  with  $\langle J \rangle \leq \langle I \rangle < N$ , which are zero. Thus  $Y_I f(a) = 0$  whenever  $\langle I \rangle < N$ . Suppose now that  $I$  is such that  $\langle I \rangle = N$  and  $X_I f(a) \neq 0$ . In the same way as in (4.2), near  $x = a$ , we may write

$$X_I = \sum_{\langle J \rangle \leq \langle I \rangle} d_{IJ}(x) Y_J,$$

for some smooth coefficients  $d_{IJ}(x)$ . Then, we have

$$0 \neq X_I f(a) = \sum_{\langle J \rangle \leq \langle I \rangle} d_{IJ}(a) Y_J f(a) = \sum_{\langle J \rangle = N} d_{IJ}(a) Y_J f(a).$$

Therefore, at least one of the numbers  $Y_J f(a)$ ,  $\langle J \rangle = N$ , must be non-zero. We then deduce that  $f$  also has order  $N$  at  $a$  with respect to the  $H$ -frame  $(Y_1, \dots, Y_n)$ . This shows that the order of  $f$  at  $a$  is independent of the choice of the  $H$ -frame. The lemma is thus proved.  $\square$

**Lemma 4.4.** *Let  $f(x)$  and  $g(x)$  be smooth functions near  $x = a$  of respective orders  $N$  and  $N'$  at  $a$ . Then  $f(x)g(x)$  has order  $\geq N + N'$  at  $a$ .*

*Proof.* We know that  $X_i(fg) = (X_i f)g + f X_i g$ . More generally, given any sequence  $I = (i_1, \dots, i_k)$ , we may write

$$(4.3) \quad X_I(fg) = X_{i_1} \cdots X_{i_k}(fg) = \sum_{\langle I' \rangle + \langle I'' \rangle = \langle I \rangle} c_{I' I''}(X_{I'} f)(X_{I''} g),$$

for some constants  $c_{IJ}$  independent of  $f$  and  $g$ . If  $\langle I' \rangle + \langle I'' \rangle < N + N'$ , then at least one of the inequalities  $\langle I' \rangle < N$  or  $\langle I'' \rangle < N'$  holds. In any case the product  $(X_{I'} f)(a)(X_{I''} g)(a)$  is zero. Combining this with (4.3) we then see that  $X_I(fg)(a) = 0$  whenever  $\langle I \rangle < N + N'$ . That is,  $f(x)g(x)$  has order  $\geq N + N'$  at  $a$ . The proof is complete.  $\square$

Given any multi-order  $\alpha \in \mathbb{N}_0^n$ , we set

$$|\alpha| = \alpha_1 + \cdots + \alpha_n \quad \text{and} \quad \langle \alpha \rangle = w_1 \alpha_1 + \cdots + w_n \alpha_n.$$

In addition, we define

$$X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}.$$

We note that  $X^\alpha = X_I$ , where  $I = (i_1, \dots, i_k)$  is the unique non-decreasing sequence of length  $k = |\alpha|$  where each index  $i$  appears with multiplicity  $\alpha_i$ . Conversely, if  $I = (i_1, \dots, i_k)$  is a non-decreasing sequence, then  $X_I = X^\alpha$  for some multi-order  $\alpha$  with  $|\alpha| = |I|$  and  $\langle \alpha \rangle = \langle I \rangle$ .

It is convenient to reformulate the definition of the order of a function in terms of the sole monomials  $X^\alpha$ . To reach this end we will make use of the following lemma.

**Lemma 4.5** (See also [Be, Lemma 4.12]). *Let  $I = (i_1, \dots, i_m)$  be a finite sequence with values in  $\{1, \dots, n\}$  and set  $w = \langle I \rangle$ . Then, near  $x = a$ , we have*

$$(4.4) \quad X_I = \sum_{\substack{\langle \alpha \rangle \leq w \\ |\alpha| \leq k}} c_{I\alpha}(x) X^\alpha,$$

where the  $c_{I\alpha}(x)$  are smooth functions near  $x = a$ .

*Proof.* We shall prove this result by induction on  $k$ . For  $k = 1$  the result is immediate. In order to prove that the results for  $k' \leq k$  imply the result for  $k + 1$  we will need the following claims.

*Claim 1.* Let  $I = (i_1, \dots, i_m)$  be a finite sequence with values in  $\{1, \dots, n\}$  and  $j$  an integer in  $\{1, \dots, n\}$ . Set  $w = \langle I \rangle + w_j$ . Then, for  $l = 1, \dots, m$  and near  $x = a$ , we may write

$$(4.5) \quad X_j X_{i_1} \cdots X_{i_m} = X_{i_1} \cdots X_{i_l} X_j X_{i_{l+1}} \cdots X_{i_m} + \sum_{\substack{\langle J \rangle \leq w \\ |J| \leq m}} c_{Ijl}^J(x) X_J,$$

for some smooth coefficients  $c_{Ijl}^J(x)$  (with the convention that  $X_{i_{l+1}} \cdots X_{i_m} = 1$  for  $l = m$ ).

*Proof of Claim 1.* We proceed by induction on  $m$ . For  $m = 1$  the claim is an immediate consequence of (4.5). Assume that the claim is true for  $m - 1$  with  $m \geq 2$  and let  $l \in \{1, \dots, m\}$ . Using (4.1) we get

$$X_j X_{i_1} \cdots X_{i_m} = X_{i_1} X_j X_{i_2} \cdots X_{i_m} + \sum_{w_p \leq w_j + w_{i_1}} L_{ji_1}^p(x) X_p X_{i_2} \cdots X_{i_m}.$$

This gives (4.5) for  $l = 1$ . If  $l \geq 2$ , then, as the claim is true for  $m - 1$ , near  $x = a$  we may write

$$(4.6) \quad X_j X_{i_2} \cdots X_{i_m} = X_{i_2} \cdots X_{i_l} X_j X_{i_{l+1}} \cdots X_{i_m} + \sum_{\substack{\langle J \rangle \leq w - w_{i_1} \\ |J| \leq m-1}} c_{Ijl}^J(x) X_J,$$

for some smooth coefficients  $c_{Ijl}^J(x)$ . Thus,

$$\begin{aligned} X_{i_1} X_j X_{i_2} \cdots X_{i_m} &= X_{i_1} X_{i_2} \cdots X_{i_l} X_j X_{i_{l+1}} \cdots X_{i_m} + \sum_{\substack{\langle J \rangle \leq w - w_{i_1} \\ |J| \leq m-1}} X_{i_1} (c_{Ijl}^J(x) X_J) \\ &= X_{i_1} X_{i_2} \cdots X_{i_l} X_j X_{i_{l+1}} \cdots X_{i_m} + \sum_{\substack{\langle J \rangle \leq w - w_{i_1} \\ |J| \leq m-1}} ((X_{i_1} c_{Ijl}^J(x))(x) + c_{Ijl}^J(x) X_{i_1}) X_J. \end{aligned}$$

Combining this with (4.6) we see that, near  $x = a$ ,

$$X_j X_{i_1} \cdots X_{i_m} = X_{i_1} \cdots X_{i_l} X_j X_{i_{l+1}} \cdots X_{i_m} + \sum_{\substack{\langle J \rangle \leq w \\ |J| \leq m}} c_{Ijl}^J(x) X_J,$$

for some smooth coefficients  $c_{Ijl}^J(x)$ . This shows that the claim is true for  $m$ . The proof of the claim is complete.  $\square$

*Claim 2.* Let  $j \in \{1, \dots, n\}$  and let  $\alpha \in \mathbb{N}_0^n$  be such that  $|\alpha| = k$ . Set  $w = w_j + \langle \alpha \rangle$ . Assume that (4.4) holds for  $|I| \leq k$ . Then there is a multi-order  $\beta$  with  $|\beta| = k + 1$  and  $\langle \beta \rangle = w$  such that, near  $x = a$ , we may write

$$X_j X^\alpha = X^\beta + \sum_{\substack{\langle \gamma \rangle \leq w \\ |\gamma| \leq k}} c_{\alpha j}^\gamma(x) X^\gamma,$$

for some smooth coefficients  $c_{\alpha j}^\gamma(x)$ .

*Proof of Claim 2.* Let  $I = (i_1, \dots, i_k)$  be the unique non-decreasing sequence of length  $k = |\alpha|$  with values in  $\{1, \dots, n\}$  such that each integer  $i$  has multiplicity  $\alpha_i$ . Note that  $\langle I \rangle = \langle \alpha \rangle$ . Let  $l_0$  be the largest integer  $l \in \{0, \dots, n\}$  such that either  $l = 0$  or  $j_l < i_1$ . Then by Claim 1, near  $x = a$ , we may write

$$(4.7) \quad X_j X^\alpha = X_j X_{i_1} \cdots X_{i_k} = X_{i_1} \cdots X_{i_l} X_j X_{i_{l+1}} \cdots X_{i_k} + \sum_{\substack{\langle J \rangle \leq w \\ |J| \leq k}} c_{Ijl}^J(x) X_J,$$

for some smooth coefficients  $c_{Ijl}^J(x)$ . As the sequence  $(i_1, \dots, i_{l_0}, j, i_{l_0+1}, \dots, i_k)$  is non-decreasing, there is a unique multi-order  $\beta \in \mathbb{N}_0^n$  with  $|\beta| = k + 1$  and  $\langle \beta \rangle = w$  such that

$$(4.8) \quad X_{i_1} \cdots X_{i_l} X_j X_{i_{l+1}} \cdots X_{i_k} = X^\beta.$$

In the summation in (4.7) all the terms  $X_J$  are of the form (4.4), since by assumption (4.4) is true for  $|I| \leq k$ . Combining this with (4.7) and (4.8) proves the claim.  $\square$

Let us go back to the proof of Lemma 4.5. We assume that (4.4) holds when  $|I| \leq k$ . Let  $I = (i_1, \dots, i_{k+1})$  be a finite sequence of length  $|I| = k + 1$ . We may apply (4.4) to  $X_{i_2} \cdots X_{i_{k+1}}$  to get

$$X_{i_2} \cdots X_{i_{k+1}} = \sum_{\substack{\langle \alpha \rangle \leq w - w_{i_1} \\ |\alpha| \leq k}} c_{I\alpha}(x) X^\alpha.$$

Using the equality  $[X_{i_1}, c_{I\alpha}] = X_{i_1}(c_{I\alpha})$ , we see that, near  $x = a$ , we have

$$\begin{aligned} X_I &= X_{i_1} X_{i_2} \cdots X_{i_{k+1}} = \sum_{\substack{\langle \alpha \rangle \leq w - w_{i_1} \\ |\alpha| \leq k}} (X_{i_1}(c_{I\alpha})(x) + c_{I\alpha}(x) X_{i_1}) X^\alpha \\ &= \sum_{\substack{\langle \alpha \rangle \leq w - w_{i_1} \\ |\alpha| \leq k}} X_{i_1}(c_{I\alpha})(x) X^\alpha + \sum_{\substack{\langle \alpha \rangle \leq w - w_{i_1} \\ |\alpha| \leq k}} c_{I\alpha}(x) X_{i_1} X^\alpha. \end{aligned}$$

Combining this with Claim 2 then shows that  $X_I$  can be put in the form (4.4). This establishes (4.4) for  $|I| = k + 1$ . The proof of Lemma 4.5 is complete.  $\square$

**Proposition 4.6.** *Let  $f(x)$  be a smooth function defined near  $x = a$ . Then  $f(x)$  has order  $N$  at  $x = a$  if and only if the following two conditions are satisfied:*

- (i)  $(X^\alpha f)(a) = 0$  for all multi-orders  $\alpha$  such that  $\langle \alpha \rangle < N$ .
- (ii)  $(X^\alpha f)(a) \neq 0$  for at least one multi-order  $\alpha$  with  $\langle \alpha \rangle = N$ .

*Proof.* Suppose that (i) and (ii) are satisfied. Then (ii) implies that  $f(x)$  has order  $\leq N$  at  $x = a$ . Moreover, using (i) and Lemma 4.5 shows that  $f(x)$  has order  $\geq N$  at  $x = a$ . Thus  $f(x)$  has order  $N$  at  $x = a$ .

Conversely, assume that  $f(x)$  has order  $N$  at  $x = a$ . It is immediate that (i) holds. Let  $I = (i_1, \dots, i_k)$  be a sequence with values in  $\{1, \dots, n\}$  with  $\langle I \rangle = N$  and  $X_I f(a) \neq 0$ . By Lemma 4.5, near  $x = a$ , we have

$$X_I = \sum_{\langle \alpha \rangle \leq \langle I \rangle} c_{I\alpha}(x) X^\alpha = \sum_{\langle \alpha \rangle \leq N} c_{I\alpha}(x) X^\alpha$$

for some smooth coefficients  $c_{I\alpha}(x)$ . Thus,

$$0 \neq X_I f(a) = \sum_{\langle \alpha \rangle \leq N} c_{I\alpha}(a) X^\alpha f(a) = \sum_{\langle \alpha \rangle = N} c_{I\alpha}(a) X^\alpha f(a).$$

This implies that at least one of the numbers  $X^\alpha f(a)$ ,  $\langle \alpha \rangle = N$ , is non-zero, i.e., (ii) is satisfied. The proof is complete.  $\square$

**Definition 4.7.** We say that local coordinates  $\{x_1, \dots, x_n\}$  centered at a point  $a \in M$  are linearly adapted at  $a$  to the  $H$ -frame  $X_1, \dots, X_n$  when  $X_j(0) = \partial_j$  for  $j = 1, \dots, n$ .



**Lemma 4.8.** *Given local coordinates  $x = (x_1, \dots, x_n)$ , there is a unique affine change of coordinates  $x \rightarrow T_a(x)$  that provides us with local coordinates centered at  $a$  that are linearly adapted to the  $H$ -frame  $(X_1, \dots, X_n)$ .*

*Proof.* In the local coordinates  $(x_1, \dots, x_n)$  we may write

$$X_i = \sum_{1 \leq j \leq n} b_{ij}(x) \partial_{x_j}, \quad i = 1, \dots, n,$$

where the coefficients  $b_{ij}(x)$  are smooth. Set  $B(x) = (b_{ij})_{1 \leq i, j \leq n} \in \text{GL}_n(\mathbb{R})$ . In what follows we shall use the same notation for the point  $a$  and its coordinate vector  $a = (a_1, \dots, a_n)$  with respect to the local coordinates  $(x_1, \dots, x_n)$ .

Let  $T(x) = A(x - a)$  be an affine transformation with  $T(a) = 0$  and  $A = (a_{jk}) \in \text{GL}_n(\mathbb{R})$ . Set  $y = (y_1, \dots, y_n) = T(x)$ , i.e.,  $y_i = \sum_j a_{ij}(x_j - a_j)$ ,  $i = 1, \dots, n$ . Then  $(y_1, \dots, y_n)$  are local coordinates centered at  $a$ . In those coordinates, we have

$$(4.9) \quad X_i = \sum_{1 \leq j, k \leq n} b_{ij}(x) \frac{\partial y_k}{\partial x_j} \frac{\partial}{\partial y_k} = \sum_{1 \leq k \leq n} \left( \sum_{1 \leq j \leq n} b_{ij}(x) a_{kj} \right) \frac{\partial}{\partial y_k}.$$

Thus  $X_i = \frac{\partial}{\partial y_k}$  at  $y = 0$  if and only if  $\sum_{1 \leq j \leq n} b_{ij}(x) a_{kj} = \delta_{ik}$ . We then see that the local coordinates  $(y_1, \dots, y_n)$  are linearly adapted at  $a$  if and only if  $B(a)A^T = 1$ , i.e.,  $A = (B(a)^T)^{-1}$ . This shows that  $T_a(x) = (B(a)^T)^{-1}(x - a)$  is the unique affine isomorphism that produces linearly adapted coordinates centered at  $a$ . The proof is complete.  $\square$

**Definition 4.9.** We say that local coordinates  $x = (x_1, \dots, x_n)$  centered at  $a$  are *privileged coordinates* at  $a$  adapted to the  $H$ -frame  $(X_1, \dots, X_n)$  when the following two conditions are satisfied:

- (i) These coordinates are linearly adapted at  $a$  to the  $H$ -frame  $(X_1, \dots, X_n)$ .
- (ii) For all  $j = 1, \dots, n$ , the coordinate function  $x_j$  has order  $w_j$  at  $a$ .

*Remark 4.10.* If the condition (i) holds, then  $X_k(x_j)(a) = \partial_k(x_j) = \delta_{jk}$ . Therefore, we see that in this case  $x_j$  has order  $w_j$  if and only if  $X^\alpha(x_j) = 0$  for all multi-orders  $\alpha$  such that  $\langle \alpha \rangle < w_j$  and  $|\alpha| \geq 2$ .

In what follows using local coordinates centered at  $a$  we may regard the vector fields  $X_1, \dots, X_n$  as vector fields defined on a neighborhood of the origin  $0 \in \mathbb{R}^n$ .

**Lemma 4.11** ([Be, Lemma 4.13]). *Let  $h(x)$  be a homogeneous polynomial of degree  $k$ . Then*

$$(X^\alpha h)(0) = \begin{cases} \partial_x^\alpha h(0) & \text{if } |\alpha| = k, \\ 0 & \text{if } |\alpha| < k. \end{cases}$$

*Remark 4.12.* In the proof of the above result in [Be, page 40], the summation in Eq. (34) is over all multi-orders  $\beta = (\beta_1, \dots, \beta_n)$  such that  $\beta \neq \alpha$  and  $\beta_i \leq \alpha_i$  for  $i = 1, \dots, n$ . This should be replaced by the summation over all multi-orders  $\beta$  such that  $|\beta| \leq |\alpha|$ .

**Proposition 4.13** (See also [Be, Lemma 4.14]). *Let  $(x_1, \dots, x_n)$  be local coordinates centered at  $a$  that are linearly adapted to the  $H$ -frame  $(X_1, \dots, X_n)$ . Then there is a unique change of coordinates  $x \rightarrow \psi(x)$  such that*

- (1) *It provides us with privileged coordinates at  $a$ .*
- (2) *For  $j = 1, \dots, n$ , the  $j$ -th component  $\psi_j(x)$  is of the form,*

$$(4.10) \quad \psi_j(x) = x_j + \sum_{\substack{\langle \alpha \rangle < w_j \\ |\alpha| \geq 2}} a_{j\alpha} x^\alpha, \quad a_{j\alpha} \in \mathbb{R}.$$

*Proof.* Let  $x \rightarrow \psi(x)$  be a change of coordinates of the form (4.10). Set  $y = \psi(x)$  and let  $j \in \{1, \dots, n\}$ . As pointed out in Remark 4.10, the coordinate  $y_j = \psi(x)_j$  has order  $w_j$  at  $a$  if

and only if  $X^\alpha(y_j)(a) = 0$  for all multi-orders  $\alpha$  such that  $\langle \alpha \rangle < w_j$  and  $|\alpha| \geq 2$ . Let  $\alpha$  be such a multi-order. Then by Lemma 4.13 we have

$$\begin{aligned} X^\alpha(y_j)|_{x=0} &= X^\alpha(x_j)|_{x=0} + \sum_{\substack{\langle \beta \rangle < w_j \\ 2 \leq |\beta|}} a_{j\beta} X^\alpha(x^\beta)|_{x=0} \\ &= X^\alpha(x_j)|_{x=0} + \sum_{\substack{\langle \beta \rangle < w_j \\ 2 \leq |\beta| < |\alpha|}} a_{j\beta} X^\alpha(x^\beta)|_{x=0} + \alpha! a_{j\alpha}. \end{aligned}$$

Thus,

$$(4.11) \quad X^\alpha(y_j)|_{x=0} = 0 \iff \alpha! a_{j\alpha} = -X^\alpha(x_j)|_{x=0} - \sum_{\substack{\langle \beta \rangle < w_j \\ 2 \leq |\beta| < |\alpha|}} a_{j\beta} X^\alpha(x^\beta)|_{x=0}.$$

As the right-hand side uniquely determines the coefficients  $a_{j\alpha}$ , we deduce that there is a unique map  $\psi(x)$  of the form (4.10) such that the change of variable  $x \rightarrow \psi(x)$  provides us with privileged coordinates at  $a$  that are linearly adapted to  $H$ -frame  $(X_1, \dots, X_n)$ . The lemma is thus proved.  $\square$

*Remark 4.14.* When  $r = 2$  the map  $\psi$  is the identity map, and so the privileged coordinates that we obtain are simply linearly adapted coordinates. In the special case of Heisenberg manifolds, these coordinates are called  $y$ -coordinates in [BG].

*Remark 4.15.* When  $r = 3$  we recover the  $p$ -coordinates of Cummins [Cu].

*Remark 4.16.* When the Carnot structure arises from an equiregular Carnot-Carathéodory structure, we recover the privileged coordinates of Bellaïche [Be].

*Remark 4.17.* It follows from the above proof that each coefficient  $a_{j\alpha}$  in (4.10) is a universal polynomial in the derivatives  $X^\alpha(x^\beta)|_{x=0}$  with  $\langle \beta \rangle \leq w_j$  and  $|\beta| \geq 1$ . Set  $X_j = \sum_{k=1}^n b_{jk}(x) \partial_k$ . An induction shows that

$$X^\alpha = \sum_{1 \leq |\beta| \leq |\alpha|} b_{\alpha\beta}(x) \partial^\beta,$$

where  $b_{\alpha\beta}(x)$  is a universal polynomial in the partial derivatives  $\partial^\gamma b_{jk}(x)$  with  $|\gamma| \leq |\alpha| - |\beta|$ . As  $X^\alpha(x^\beta)|_{x=0} = \beta! b_{\alpha\beta}(0)$ , we then deduce that each coefficient  $a_{j\alpha}$  is a universal polynomial in the partial derivatives  $\partial^\gamma b_{kl}(0)$  with  $|\gamma| \leq |\alpha| - 1$ .

**Definition 4.18.** Let  $(x_1, \dots, x_n)$  be the linearly adapted coordinates provided by the affine map  $T_a$  from Lemma 4.8. Then we denote by  $\psi_a(x)$  the polynomial diffeomorphism associated with these coordinates that is provided by Proposition 4.13.

We conclude this section with the following uniqueness result.

**Proposition 4.19.** *The coordinates  $y = \psi_a(T_a x)$  are the unique privileged coordinates at  $a$  adapted to the  $H$ -frame  $(X_1, \dots, X_n)$  that are given by a change of coordinates of the form  $y = \psi(T(x))$ , where  $T$  is an affine map such that  $T(a) = 0$  and  $\psi(x)$  is a polynomial diffeomorphism of the form (4.10).*

*Proof.* Let  $x \rightarrow \phi(x)$  be a change of coordinates providing us with privileged coordinates at  $a$  adapted to the  $H$ -frame  $(X_1, \dots, X_n)$  such that  $\phi(x) = \psi(T(x))$ , where  $T$  is an affine map such that  $T(a) = 0$  and  $\psi(x)$  is a polynomial diffeomorphism of the form (4.10). As privileged coordinates are linearly adapted coordinates, we see that  $\phi_* X_j(0) = \partial_j$  for  $j = 1, \dots, n$ . Note that (4.10) implies that  $\psi'(0) = \text{id}$ . Thus  $\phi_* X_j(0) = \psi'(0) \circ T'(a)(X_j(a)) = T'(a)(X_j(a)) = T_* X_j(0)$ , so that we see that  $T_* X_j(0) = \partial_j$ . This means that the coordinate change  $x \rightarrow T(x)$  provides us with coordinates that are linearly adapted at  $a$  to the  $H$ -frame  $(X_1, \dots, X_n)$ . As  $T(x)$  is an affine map, it then follows from Lemma 4.8 that  $T(x) = T_a(x)$ . Therefore, we see that  $\psi(x)$  is a polynomial diffeomorphism of the form (4.10) that transforms the coordinates  $y = \psi_a(x)$  into privileged coordinates at  $a$  adapted to  $(X_1, \dots, X_n)$ . It then follows from the uniqueness contents of Proposition 4.13 that  $\psi(x) = \psi_a(x)$ , so that  $\phi(x) = \psi_a(T_a x)$ . This proves the result.  $\square$

## 5. NILPOTENT APPROXIMATION

In this section, we explain how privileged coordinates at a given point enables us to obtain an extrinsic construction of a tangent group at that point by means of the “nilpotent approximation” of vector fields. In particular, we give a very precise sense to the various asymptotics that occur in the process. Nilpotent approximations have been considered by various authors [BG, Be, RS, Go, He, Ro], often by working with a specific type of privileged coordinates. We shall give here a systematic account for all types of privileged coordinates, including a precise description of the various asymptotics at stake.

**5.1. Weighted homogeneous approximation.** As in Section 2.1, we denote by  $\delta_t$ ,  $t > 0$ , the (anisotropic) dilations on  $\mathbb{R}^n$  defined by

$$(5.1) \quad \delta_t(x) := t \cdot x := (t^{w_1}x_1, \dots, x_n), \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

**Definition 5.1.** A function  $f(x)$  on  $\mathbb{R}^n$  or  $\mathbb{R} \setminus 0$  is *homogeneous* of degree  $w$ ,  $w \in \mathbb{R}$ , with respect to the dilations (5.1) when

$$f(t \cdot x) = t^w f(x) \quad \text{for all } t > 0.$$

*Example 5.2.* For any multi-order  $\alpha \in \mathbb{N}_0^n$ , the monomial  $x^\alpha$  is homogeneous of degree  $\langle \alpha \rangle$ .

*Remark 5.3.* If  $f(x)$  is smooth and homogeneous of degree  $w$ , then differentiating the equality  $f(t \cdot x) = t^w f(x)$  shows that  $\partial^\alpha f(t \cdot x) = t^{w - \langle \alpha \rangle} \partial^\alpha f(x)$ . Thus  $\partial^\alpha f(x)$  is homogeneous of degree  $w - \langle \alpha \rangle$ . If we further assume that  $f(x)$  is smooth at  $x = 0$  and we choose  $\alpha$  so that  $\langle \alpha \rangle > w$ , then, as  $t \rightarrow 0^+$ , we have

$$\partial^\alpha f(x) = t^{\langle \alpha \rangle - w} \partial^\alpha f(t \cdot x) \longrightarrow 0 \cdot \partial^\alpha f(0) = 0.$$

Therefore, all the partial derivatives  $\partial^\alpha f$  with  $\langle \alpha \rangle > w$  are identically zero. Combining this with the inequality  $\langle \alpha \rangle < r|\alpha|$  shows that  $\partial^\alpha f(x)$  is identically zero as soon as  $|\alpha|$  is large enough. It then follows that  $f(x)$  is a polynomial function and  $w$  must be a non-negative integer.

In what follows we let  $U$  be an open neighborhood of the origin  $0 \in \mathbb{R}^n$ . We mention the following version of Taylor’s formula.

**Lemma 5.4** (Weighted Taylor’s Formula). *Let  $f \in C^\infty(U)$  and  $N \in \mathbb{N}$ . Then there are functions  $R_{N\alpha} \in C^\infty(U)$ ,  $|\alpha| \leq N \leq \langle \alpha \rangle$ , such that*

$$(5.2) \quad f(x) = \sum_{\langle \alpha \rangle < N} \frac{1}{\alpha!} \partial_x^\alpha f(0) x^\alpha + \sum_{|\alpha| \leq N \leq \langle \alpha \rangle} x^\alpha R_{N\alpha}(x).$$

*Proof.* By Taylor’s formula there are functions  $R_{N\alpha} \in C^\infty(U)$ ,  $|\alpha| = N$ , such that

$$f(x) = \sum_{|\alpha| < N} \frac{1}{\alpha!} \partial_x^\alpha f(0) x^\alpha + \sum_{|\alpha| = N} x^\alpha R_{N\alpha}(x).$$

Using the inequality  $|\alpha| \leq \langle \alpha \rangle$  we may rewrite this as

$$f(x) = \sum_{\langle \alpha \rangle < N} \frac{1}{\alpha!} \partial_x^\alpha f(0) x^\alpha + \sum_{|\alpha| < N \leq \langle \alpha \rangle} \frac{1}{\alpha!} \partial_x^\alpha f(0) x^\alpha + \sum_{|\alpha| = N \leq \langle \alpha \rangle} x^\alpha R_{N\alpha}(x).$$

Setting  $R_{N\alpha}(x) = \frac{1}{\alpha!} \partial_x^\alpha f(0) x^\alpha$  for  $|\alpha| < N \leq \langle \alpha \rangle$  then gives an expansion of the form (5.2). The proof is complete.  $\square$

**Definition 5.5.** Let  $f \in C^\infty(U)$ . We shall say that

- (1)  $f$  has weight  $\geq w$  when  $\partial_x^\alpha f(0) = 0$  for all multi-orders  $\alpha \in \mathbb{N}_0^n$  such that  $\langle \alpha \rangle < w$ .
- (2)  $f$  has weight  $w$  when  $f(x)$  has weight  $\geq w$  and there is a multi-order  $\alpha \in \mathbb{N}_0^n$  with  $\langle \alpha \rangle = w$  such that  $\partial_x^\alpha f(0) \neq 0$ .

In the following we equip  $C^\infty(U)$  with its standard locally convex space topology, i.e., the LCS topology defined by the semi-norms,

$$p_{K,\alpha}(f) = \sup_{x \in K} |\partial^\alpha f(x)|, \quad \alpha \in \mathbb{N}_0^n, \quad K \subset U \text{ compact.}$$

Therefore, a sequence  $(f_\ell)_\ell \subset C^\infty(U)$  converges to a function  $f$  in  $C^\infty(U)$  if and only if, for all multi-orders  $\alpha \in \mathbb{N}_0^n$ , the partial derivatives  $\partial^\alpha f_\ell$  converge to  $\partial^\alpha f$  uniformly on compact subsets of  $U$ .

Note that the semi-norm  $p_{K,\alpha}(f)$  actually makes sense for any smooth function on an open neighborhood of  $K$ . We also observe that a basis of neighborhoods of the origin in  $\mathbb{R}^n$  is provided by the anisotropic balls,

$$B(\rho) = \{x \in \mathbb{R}^n; \|x\| < \rho\}, \quad \rho > 0.$$

In particular, there is  $\rho_0 > 0$  such that  $B(\rho_0) \subset U$ . We also note that if  $K$  is a compact subset of  $U$ , then there is  $t_0 > 0$  such that  $\delta_t(K)$  is contained in  $B(\rho_0)$  for all  $t \in [0, t_0]$ . Therefore, if  $f \in C^\infty(U)$ , then the semi-norm  $p_{K,\alpha}(f \circ \delta_t)$  makes sense as soon as  $t$  is sufficiently small.

**Lemma 5.6.** *Let  $f \in C^\infty(U)$ . Then the following are equivalent:*

- (1) *The function  $f$  has weight  $w$ .*
- (2) *As  $t \rightarrow 0^+$ , we have*

$$(5.3) \quad f(t \cdot x) \simeq \sum_{l \geq w} t^l f^{[l]}(x) \quad \text{in } C^\infty(U),$$

where  $f^{[l]}(x)$  is a homogeneous polynomial of degree  $l$  with  $f^{[w]} \neq 0$ .

*Remark 5.7.* The fact that the asymptotic expansion (5.3) holds in  $C^\infty(U)$  means that, for all  $\alpha \in \mathbb{N}_0^n$  and  $N > w$  and for all compact  $K \subset U$ , we have

$$p_{K,\alpha} \left( f \circ \delta_t - \sum_{w \leq l < N} t^l f^{[l]} \right) = O(t^N) \quad \text{as } t \rightarrow 0^+.$$

*Proof of Lemma 5.6.* Let  $N \in \mathbb{N}$ . By Lemma 5.4 there are functions  $R_{N\alpha} \in C^\infty(U)$ ,  $|\alpha| \leq N \leq \langle \alpha \rangle$ , such that

$$f(x) = \sum_{\langle \alpha \rangle < N} \frac{1}{\alpha!} \partial_x^\alpha f(0) x^\alpha + \sum_{|\alpha| \leq N \leq \langle \alpha \rangle} x^\alpha R_{N\alpha}(x).$$

As the monomial  $x^\alpha$  is homogeneous of degree  $\langle \alpha \rangle$ , we get

$$f(t \cdot x) = \sum_{\langle \alpha \rangle < N} t^{\langle \alpha \rangle} \frac{1}{\alpha!} \partial_x^\alpha f(0) x^\alpha + \sum_{|\alpha| \leq N \leq \langle \alpha \rangle} t^{\langle \alpha \rangle} x^\alpha R_{N\alpha}(t \cdot x).$$

The smoothness of  $R_{N\alpha}(x)$  implies that  $R_{N\alpha}(t \cdot x)$  is  $O(1)$  in  $C^\infty(U)$  as  $t \rightarrow 0^+$ . Therefore, we see that, as  $t \rightarrow 0^+$  and in  $C^\infty(U)$ , we have

$$(5.4) \quad f(t \cdot x) = \sum_{\langle \alpha \rangle < N} t^{\langle \alpha \rangle} \frac{1}{\alpha!} \partial_x^\alpha f(0) x^\alpha + O(t^N).$$

For  $l = 0, 1, \dots$  set  $f^{[l]}(x) = \sum_{\langle \alpha \rangle = l} \frac{1}{\alpha!} \partial_x^\alpha f(0) x^\alpha$ . Then  $f^{[l]}(x)$  is a homogeneous polynomial of degree  $l$ . Moreover, Eq. (5.4) shows that in  $C^\infty(U)$  we have

$$f(t \cdot x) \simeq \sum_{l \geq 0} t^l f^{[l]}(x) \quad \text{as } t \rightarrow 0^+.$$

This is an asymptotic expansion of the form (5.3) if and only if  $f^{[l]} = 0$  for  $l < w$  and  $f^{[w]} \neq 0$ . The latter condition exactly means that  $\partial_x^\alpha f(0) = 0$  for  $\langle \alpha \rangle < w$  and  $\partial_x^\alpha f(0) \neq 0$  for some multi-order  $\alpha$  with  $\langle \alpha \rangle = w$ , i.e.,  $f$  has weight  $w$ . This shows that  $f$  has weight  $w$  if and only if it admits an asymptotic expansion of the form (5.3). The lemma is thus proved.  $\square$

The notion of weight of a function extends to differential operators as follows. Given a differential operator  $P$  on  $U$ , for  $t > 0$  we denote by  $\delta_t^* P$  the pullback of  $P$  by the dilation  $\delta_t$ , i.e.,

$$(\delta_t^* P)u(x) = P(u \circ \delta_t^{-1})(t \cdot x) \quad \text{for all } u \in C^\infty(\delta_t(U)).$$

**Definition 5.8.** A differential operator  $P$  on  $\mathbb{R}^n$  or  $\mathbb{R} \setminus 0$  is *homogeneous* of degree  $w$ ,  $w \in \mathbb{R}$ , when

$$\delta_t^* P = t^w P \quad \text{for all } t > 0.$$

*Example 5.9.* For any multi-order  $\alpha$ , the differential operator  $\partial^\alpha$  is homogeneous of degree  $-\langle \alpha \rangle$ .

**Definition 5.10.** Let  $P = \sum_{|\alpha| \leq m} a_\alpha(x) \partial_x^\alpha$  be a differential operator on  $U$ . We say that  $P$  has weight  $w$ ,  $w \in \mathbb{R}$ , when

- (1) Each coefficient  $a_\alpha(x)$  has weight  $\geq w + \langle \alpha \rangle$ .
- (2) There is one coefficient  $a_\alpha(x)$  that has weight  $w + \langle \alpha \rangle$ .

In what follows, for  $m \geq \mathbb{N}$ , we denote by  $\text{DO}^m(U)$  the space of  $m$ -th order differential operators on  $U$ . If  $P$  is an operator in  $\text{DO}^m(U)$ , then are unique coefficients  $a_\alpha(P)(x) \in C^\infty(U)$ ,  $|\alpha| \leq m$ , such that

$$P = \sum_{|\alpha| \leq m} a_\alpha(P)(x) \partial^\alpha.$$

We then have linear maps  $\text{DO}^m(U) \ni P \rightarrow a_\alpha(P) \in C^\infty(U)$ . We then equip  $\text{DO}^m(U)$  with the coarsest topology with respect to which all these maps are continuous. Therefore, as an LCS  $\text{DO}^m(U)$  is naturally isomorphic to  $C^\infty(U)^{N(m)}$ , where  $N(m) = \#\{\alpha \in \mathbb{N}_0^n; |\alpha| \leq m\}$ .

**Lemma 5.11.** Let  $P \in \text{DO}^m(U)$  and assume that not all the coefficients of  $P$  vanish at  $x = 0$ . Then the following are equivalent:

- (1) The differential operator  $P$  has weight  $w$ .
- (2) As  $t \rightarrow 0^+$ , we have

$$(5.5) \quad \delta_t^* P \simeq \sum_{l \geq w} t^l P^{[l]} \quad \text{in } \text{DO}^m(U),$$

where  $P^{[l]}$  is a homogeneous differential operator of degree  $l$  with  $P^{[w]} \neq 0$ .

*Proof.* Let us write  $P = \sum_{|\alpha| \leq m} a_\alpha(x) \partial_x^\alpha$ , with  $a_\alpha \in C^\infty(U)$ . Then

$$\delta_t^* P = \sum_{|\alpha| \leq m} a_\alpha(t \cdot x) \delta_t^* \partial_x^\alpha = \sum_{|\alpha| \leq m} t^{-w_j} a_j(t \cdot x) \partial_x^\alpha.$$

For  $\alpha \in \mathbb{N}_0^N$ ,  $|\alpha| \leq m$ , let us denote by  $w(\alpha)$  the weight of the function  $a_\alpha(x)$ . Using Lemma 5.6 we see that, as  $t \rightarrow 0^+$ , we have

$$(5.6) \quad \delta_t^* P \simeq \sum_{|\alpha| \leq m} \sum_{l_\alpha \geq w(\alpha)} t^{l_\alpha - w_\alpha} a_\alpha^{[l_\alpha]}(x) \partial_x^\alpha \quad \text{in } \text{DO}^m(U).$$

We note that  $a_\alpha^{[l_\alpha]} \partial_x^\alpha$  is a homogeneous differential operator of degree  $l_\alpha - w_\alpha$ . Therefore, the asymptotic expansion (5.6) is of the form (5.5) if and only if  $w(\alpha) - w_\alpha \geq w$  for all multi-orders  $\alpha$  with  $|\alpha| \leq m$  and there is one such multi-order such that  $w(\alpha) - w_\alpha = w$ . The latter conditions mean that each coefficient  $a_\alpha(x)$  has weight  $\geq w + w_j$  and there is equality for at least one of those, that is, the differential operator  $P$  has weight  $w$ . Therefore, we see that  $P$  has weight  $w$  if and only if it admits an asymptotic expansion of the form (5.5). The lemma is thus proved.  $\square$

*Remark 5.12.* In the above proof we have  $a_\alpha^{[l]}(x) = \sum_{\langle \beta \rangle = l} \frac{x^\beta}{\beta!} \partial^\beta a_\alpha(0)$ . Therefore, we see that the weight  $w$  of  $P$  is given by

$$w = \min\{\langle \beta \rangle - \langle \alpha \rangle; \partial^\beta a_\alpha(0) \neq 0\}.$$

Thus,

$$\min\{-\langle \alpha \rangle; |\alpha| \leq m\} \leq w \leq \min\{-\langle \alpha \rangle; a_\alpha(0) \neq 0\}$$

In particular, the weight  $w$  is always a negative integer.

*Remark 5.13.* The asymptotic expansion (5.5) implies that

$$\lim_{t \rightarrow 0} t^{-w} \delta_t^* P = P^{[w]} \quad \text{in } \text{DO}^m(U).$$

Let us now specialize the above results to vector fields. Let  $\mathcal{X}(U)$  be the space of vector fields on  $U$ . Regarding it as a (closed) subspace of  $\text{DO}^1(U)$  we equip it with the induced topology. Equivalently, a vector field  $X$  on  $U$  has a unique expression as

$$X = \sum_{1 \leq j \leq n} a_j(X)(x) \partial_j, \quad \text{with } a_j(X) \in C^\infty(U).$$

The topology on  $\mathcal{X}(U)$  then is the coarsest topology with respect to which all the linear maps  $\mathcal{X}(U) \ni X \rightarrow a_j(X) \in C^\infty(U)$ ,  $j = 1, \dots, n$ , are continuous. Incidentally, as an LCS  $\mathcal{X}(U)$  is naturally isomorphic to  $C^\infty(U)^n$ .

Specializing Lemma 5.11 to vector fields gives the following statement.

**Lemma 5.14.** *Let  $X \in \mathcal{X}(U)$  be such that  $X(0) \neq 0$ . Then the following are equivalent:*

- (1)  *$X$  has weight  $w$ .*
- (2) *As  $t \rightarrow 0^+$ , we have*

$$(5.7) \quad \delta_t^* X \simeq \sum_{l \geq w} t^l X^{[l]} \quad \text{in } \mathcal{X}(U),$$

where  $X^{[l]}$  is a homogeneous vector field of degree  $l$  with  $X^{[w]} \neq 0$ .

*Remark 5.15.* The weight of  $X$  is given by

$$w = -\max\{w_j; a_j(X)(0) \neq 0\}.$$

In particular, this is always a negative integer between  $-r$  and  $-1$ .

**5.2. Weight and privileged coordinates.** Let  $a \in M$  and  $(x_1, \dots, x_n)$  local coordinates centered at  $a$ . We denote by  $U$  the range of these local coordinates and by  $V$  their domain. Note that  $U$  is an open neighborhood of the origin in  $\mathbb{R}^n$ . We shall further assume there is an  $H$ -frame over  $V$ .

Using the local coordinates  $(x_1, \dots, x_n)$  we can regard any function (resp., vector field, differential operator) on  $V$  as a function (resp., vector field, differential operator) on  $U$ . This enables us to define the *weight* of such objects. In fact, as the weight depends only on the germ near  $x = 0$ , we actually can define the weight for any such object defined near  $x = a$ . It should be however pointed out this notion of weight is *extrinsic* as it depends on the choice of the local coordinates. For instance, it is not preserved by permutation of the coordinates  $(x_1, \dots, x_n)$  (unless  $r = 1$ ). For this reason we will refer to this weight as the weight in the local coordinates  $(x_1, \dots, x_n)$ .

**Definition 5.16.** Let  $X$  be a vector field on an open neighborhood of  $a$ . Let  $w$  be its weight in the local coordinates  $(x_1, \dots, x_n)$ . Then the vector field  $X^{[w]}$  in (5.7) is denoted by  $X^{(a)}$  and is called the *model vector field* of  $X$  in the local coordinates  $(x_1, \dots, x_n)$ .

*Remark 5.17.* The asymptotic expansion (5.7) implies that

$$(5.8) \quad \lim_{t \rightarrow 0} t^{-w} \delta_t^* X = X^{(a)} \quad \text{in } \mathcal{X}(U).$$

*Remark 5.18.* We similarly can define a notion of model operator for any differential operator on an open neighborhood of  $a$ .

As mentioned above, the notion of weight is an *extrinsic* notion. However, as the following shows, when using privileged coordinates this extrinsic notion of weight actually agrees with the *intrinsic* notion of order defined in the previous section.

**Lemma 5.19.** *Assume that the coordinates  $(x_1, \dots, x_n)$  are privileged coordinates at  $a$  adapted to the  $H$ -frame  $(X_1, \dots, X_n)$ . Let  $f(x)$  be a smooth function on a neighborhood of  $a$ . Then its weight in the local coordinates  $(x_1, \dots, x_n)$  agrees with its order at  $a$ .*



*Proof.* Let  $w$  be the weight of  $f$ . As its order is an intrinsic notion, we may work in the local coordinates  $(x_1, \dots, x_n)$ . Note that as they are privileged coordinates at  $a$ , for  $j = 1, \dots, n$ , the coordinate  $x_j$  has order  $w_j$ . Moreover, the fact that  $f(x)$  has weight  $w$  implies that  $\partial^\beta f(0) = 0$  for  $\langle \beta \rangle < w$ . Therefore, applying Lemma 5.4 to  $f(x)$  and  $N = w + 1$  shows there are functions  $R_\beta \in C^\infty(U)$ ,  $|\beta| \leq w + 1 \leq \langle \beta \rangle$ , such that

$$f(x) = \sum_{\langle \beta \rangle = w} \frac{1}{\beta!} \partial^\beta f(0) x^\beta + \sum_{|\beta| \leq w+1 \leq \langle \beta \rangle} x^\beta R_\beta(x).$$

It follows from Lemma 4.4 that each monomial  $x^\beta$  has order  $\langle \beta \rangle$  and each remainder term  $x^\beta R_\beta(x)$ ,  $|\beta| \leq w + 1 \leq \langle \beta \rangle$ , has order  $\geq w + 1$ . Therefore, we see that  $f(x)$  is a linear combination of functions of order  $\geq w$ , and hence has order  $\geq w$ .

As  $f(x)$  has weight  $w$ , there is  $\alpha \in \mathbb{N}_0^n$  such that  $\langle \alpha \rangle = w$  and  $\partial^\alpha f(0) \neq 0$  and  $\partial^\beta f(0) = 0$  for  $|\beta| > |\alpha|$ . The fact that  $x^\beta R_\beta(x)$ ,  $|\beta| \leq w + 1 \leq \langle \beta \rangle$ , has order  $\geq w + 1$ , implies that  $X^\alpha(x^\beta R_\beta)(0) = 0$ . Therefore, using Lemma 4.11 we get

$$X^\alpha f(0) = \sum_{\substack{\langle \beta \rangle = w \\ |\beta| \leq |\alpha|}} \frac{1}{\beta!} \partial^\beta f(0) X^\alpha(x^\beta)(0) = \sum_{\substack{\langle \beta \rangle = w \\ |\beta| = |\alpha|}} \frac{1}{\beta!} \partial^\beta f(0) \partial^\alpha(x^\beta)(0) = \partial^\alpha f(0) \neq 0.$$

Using Proposition 5.20 we then deduce that  $f(x)$  has order  $w$ . The lemma is thus proved.  $\square$

We are now in a position to establish the following characterization of privileged coordinates.

**Proposition 5.20.** *Let  $(x_1, \dots, x_n)$  be local coordinates centered at  $a$  that are linearly adapted to the  $H$ -frame  $(X_1, \dots, X_n)$ . Then the following are equivalent*

- (i) *The local coordinates  $(x_1, \dots, x_n)$  are privileged coordinates at  $a$ .*
- (ii) *For  $j = 1, \dots, n$ , the vector field  $X_j$  has weight  $-w_j$  in the local coordinates  $(x_1, \dots, x_n)$ .*
- (iii) *For every multi-order  $\alpha \in \mathbb{N}_0^n$ , the differential operator  $X^\alpha$  has weight  $-\langle \alpha \rangle$  in the local coordinates  $(x_1, \dots, x_n)$ .*

Moreover, if  $X_j$  has weight  $-w_j$ , then its model vector field in the local coordinates  $(x_1, \dots, x_n)$  is of the form,

$$(5.9) \quad X_j^{(a)} = \partial_j + \sum_{\substack{w_k - \langle \alpha \rangle = w_j \\ |\alpha| \geq 1}} b_{jk\alpha} x^\alpha \partial_k, \quad b_{jk\alpha} \in \mathbb{R}.$$

*Proof.* Note that if  $X = \sum a_j(x) \partial_j$  is a vector field near the origin, then  $X(x_j) = \sum a_k(x) \partial_k(x_j) = a_j(x)$  for  $j = 0, \dots, n$ . More generally, if  $P = \sum a_\alpha(x) \partial^\alpha$  is a differential operator near the origin such that  $P(0) = 0$ , then  $a_\alpha(x) = P(x^\alpha)$  for  $|\alpha| = 1$ . Bearing this in mind, assume that  $(x_1, \dots, x_n)$  are privileged coordinates at  $a$ . The coefficient of  $\partial_k$  of  $X_j$  is  $X_j(x_k)$  and has order  $w_k - w_j$ , and hence has weight  $w_k - w_j$  by Lemma 5.19. It then follows that  $X_j$  has weight  $-w_j$ . This shows that (i) implies (ii).

Suppose that for  $j = 1, \dots, n$  the vector field  $X_j$  has weight  $-w_j$ . Set  $X_j = \sum a_{jk}(x) \partial_k$  with  $a_{jk}(x) \in C^\infty(U)$ . As the coordinates  $(x_1, \dots, x_n)$  are linearly adapted at  $a$  to the  $H$ -frame  $(X_1, \dots, X_n)$ . We have  $a_{jk}(0) = \delta_{jk}$  for  $j, k = 1, \dots, n$ . Therefore, the model vector field of  $X_j$  is given by

$$(5.10) \quad X_j^{(a)} = \sum_{w_k - \langle \alpha \rangle = w_j} \frac{1}{\alpha!} \partial^\alpha a_{jk}(0) x^\alpha \partial_k = \partial_j + \sum_{\substack{w_k - \langle \alpha \rangle = w_j \\ |\alpha| \geq 1}} \frac{1}{\alpha!} \partial^\alpha a_{jk}(0) x^\alpha \partial_k.$$

This proves (5.9). In addition, let  $\alpha \in \mathbb{N}_0^n$  and set  $m = |\alpha|$ . Using (5.8) we deduce that, as  $t \rightarrow 0^+$ , we have

$$t^{\langle \alpha \rangle} \delta_t^* X^\alpha = (t^{w_1} \delta_t^* X_1)^{\alpha_1} \dots (t^{w_n} \delta_t^* X_n)^{\alpha_n} \longrightarrow \left(X_1^{(a)}\right)^{\alpha_1} \dots \left(X_n^{(a)}\right)^{\alpha_n} \quad \text{in } \text{DO}^m(U).$$

As (5.10) implies that  $\left(X_1^{(a)}\right)^{\alpha_1} \dots \left(X_n^{(a)}\right)^{\alpha_n} \neq 0$ , this shows that  $X^\alpha$  has weight  $-\langle \alpha \rangle$ . Thus (ii) implies (iii).

It remains to show that (iii) implies (i). Suppose that, for every  $\alpha \in \mathbb{N}_0^n$ , the differential operator  $X^\alpha$  has weight  $-\langle \alpha \rangle$ . We observe that as  $X^\alpha$  is a product of vector fields we have  $X^\alpha(0) = 0$ . Therefore, by the observation made at the beginning of the proof, the coefficient of  $\partial_j$  is equal to  $X^\alpha(x_j)$ . The fact that  $X^\alpha$  has weight  $-\langle \alpha \rangle$  then implies that  $X^\alpha(x_j)$  has weight  $\geq w_j - \langle \alpha \rangle$ , and hence vanishes at  $x = 0$  if  $\langle \alpha \rangle < w_j$ . Therefore, the coordinate  $x_j$  has order  $\geq w_j$  for  $j = 1, \dots, n$ . Using Remark 4.10 we then deduce that the local coordinates  $(x_1, \dots, x_n)$  are privileged coordinates. This shows that (iii) implies (i). The proof is complete.  $\square$

**5.3. The extrinsic tangent group.** We shall now explain how the previous consideration enables us to get another notion of tangent group.

**Definition 5.21.** Given privileged coordinates at  $a$  adapted to the  $H$ -frame  $(X_1, \dots, X_n)$ , we denote by  $\mathfrak{g}^{(a)}$  the subspace of  $T\mathbb{R}^n$  spanned by the homogeneous vector fields  $X_j^{(a)}$ ,  $j = 1, \dots, n$ .

For  $w = 1, \dots, r$ , let us denote by  $\mathfrak{g}_w^{(a)}$  the subspace of  $\mathfrak{g}^{(a)}$  spanned by vector fields  $X_j^{(a)}$ ,  $w_j = w$ . This provides us with the grading,

$$(5.11) \quad \mathfrak{g}^{(a)} = \mathfrak{g}_1^{(a)} \oplus \dots \oplus \mathfrak{g}_r^{(a)}.$$

Moreover, as  $(X_1, \dots, X_n)$  is an  $H$ -frame, there are smooth functions  $L_{ij}^k(x)$ ,  $w_k \leq w_i + w_j$ , defined near  $a$  such that, for  $i, j = 1, \dots, n$ , we have

$$(5.12) \quad [X_j, X_k] = \sum_{w_k \leq w_i + w_j} L_{ij}^k(x) X_k.$$

**Lemma 5.22.** For  $i, j = 1, \dots, n$ , we have

$$(5.13) \quad [X_i^{(a)}, X_j^{(a)}] = \begin{cases} \sum_{w_k = w_i + w_j} L_{ij}^k(a) X_k^{(a)} & \text{if } w_i + w_j \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* In  $\mathcal{X}(U)$  we have

$$[X_i^a, X_j^a] = \lim_{t \rightarrow 0} [t^{w_i} \delta_t^* X_i, t^{w_j} \delta_t^* X_j] = \lim_{t \rightarrow 0} t^{w_i + w_j} \delta_t^* [X_i, X_j].$$

Combining this with (5.12) we get

$$[X_i^a, X_j^a] = \sum_{w_k \leq w_i + w_j} \lim_{t \rightarrow 0} t^{w_i + w_j} \delta_t^* (L_{ij}^k X_k) = \sum_{w_k \leq w_i + w_j} L_{ij}^k(a) \lim_{t \rightarrow 0} t^{w_i + w_j} \delta_t^* X_k.$$

Note that  $\lim_{t \rightarrow 0} t^{w_i + w_j} \delta_t^* X_k = X_k^a$  if  $w_k = w_i + w_j$  and  $\lim_{t \rightarrow 0} t^{w_i + w_j} \delta_t^* X_k = 0$  if  $w_k < w_i + w_j$ . Therefore,  $[X_i^a, X_j^a]$  is equal to  $\sum_{w_k = w_i + w_j} L_{ij}^k(a) X_k^{(a)}$  if  $w_i + w_j \leq r$  and is zero otherwise. The proof is complete.  $\square$

As an immediate consequence of Lemma 5.22 we obtain the following result.

**Proposition 5.23.** With respect to the Lie bracket of vector fields and the grading (5.11) the vector space  $\mathfrak{g}^{(a)}$  is a step  $r$  Carnot-type algebra.

In fact, it follows from (2.12) and Lemma 5.22 that the Lie algebras  $\mathfrak{g}M(a)$  and  $\mathfrak{g}^{(a)}$  have the same structure constants with respect to their respective bases  $\{\dot{X}_j(a)\}$  and  $\{X_j^{(a)}\}$ . Therefore, we arrive at the following statement.

**Proposition 5.24.** Given privileged coordinates at  $a$  relatively to the  $H$ -frame  $(X_1, \dots, X_n)$ , let  $\xi_a : \mathfrak{g}M(a) \rightarrow \mathfrak{g}^{(a)}$  be the linear map defined by

$$(5.14) \quad \xi_a(x_1 \dot{X}_1(a) + \dots + x_n \dot{X}_n(a)) = x_1 X_1^{(a)} + \dots + x_n X_n^{(a)} \quad \forall x_j \in \mathbb{R}.$$

Then  $\xi_a$  is a Carnot-type Lie algebra isomorphism from  $\mathfrak{g}M(a)$  onto  $\mathfrak{g}^{(a)}$ .

We also observe that (5.13) implies that, for  $i, j = 1, \dots, n$ , the vector field  $[X_j^{(a)}, X_j^{(a)}]$  is homogeneous of degree  $-(w_i + w_j)$ . Therefore, for all  $t > 0$ , we have

$$\delta_t^*[X_i^{(a)}, X_j^{(a)}] = t^{-(w_i + w_j)}[X_i^{(a)}, X_j^{(a)}] = [\delta_t^* X_i^{(a)}, \delta_t^* X_j^{(a)}].$$

It then follows that the dilations  $\delta_t^*$ ,  $t > 0$ , induce Lie algebra automorphisms of  $\mathfrak{g}^{(a)}$ . These dilations actually define a grading on  $\mathfrak{g}^{(a)}$  which agrees with the Carnot grading (5.11). Namely, for  $w = 1, \dots, r$ , we have

$$g_w^{(a)} = \left\{ X \in \mathfrak{g}^{(a)}; \delta_t^* X = t^{-w} X \text{ for all } t > 0 \right\}.$$

As  $\mathfrak{g}^{(a)}$  is a Lie algebra of vector fields, it is natural to realize it as a Lie algebra of left-invariant vector fields on a nilpotent Lie group  $G^{(a)}$ . We realize  $G^{(a)}$  as  $\mathbb{R}^n$  with exponential map,

$$(5.15) \quad \exp^{(a)} : \mathfrak{g}^{(a)} \ni X \longrightarrow \exp(X), \quad \exp(X) := \exp(tX)(0)|_{t=1},$$

where  $t \rightarrow \exp(tX)$  is the flow of the vector field  $X \in \mathfrak{g}^{(a)}$ . The product of  $G^{(a)}$  is obtained by using the Baker-Campbell-Hausdorff formula. Namely, given  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  in  $\mathbb{R}^n$  the product  $x \cdot y$  is given by

$$(5.16) \quad \begin{aligned} x \cdot y &= \exp \left( X + \left( \int_0^1 \Phi(e^{\text{ad}_X} e^{s \text{ad}_Y} - I) ds \right) Y \right), \\ &= \exp \left( X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \dots \right), \end{aligned}$$

where  $\Phi$  is defined by (2.14) and we have set  $X = x_1 X_1^{(a)} + \dots + x_n X_n^{(a)}$  and  $Y = y_1 X_1^{(a)} + \dots + y_n X_n^{(a)}$ . Note also that  $G^{(a)}$  is a Carnot-type group since  $\mathfrak{g}^{(a)}$  is a Carnot-type Lie algebra.

**Definition 5.25.** The Carnot-type group  $G^{(a)}$  (resp., its Lie algebra  $\mathfrak{g}^{(a)}$ ) is called the extrinsic tangent group (resp., extrinsic tangent Lie algebra) at  $a$  with respect to the privileged coordinates  $(x_1, \dots, x_n)$ .

We observe that, for any  $t > 0$  and  $X \in \mathfrak{g}^{(a)}$ , we have

$$(5.17) \quad t^{-1} \cdot \exp(X) = \delta_{t^{-1}} \circ \exp(sX) \circ \delta_t(0)|_{s=1} = \exp(\delta_t^* X).$$

As the operators  $\delta_t^*$ ,  $t > 0$ , are Lie algebra automorphisms of  $\mathfrak{g}^{(a)}$ , we deduce that the dilations  $\delta_t$ ,  $t > 0$ , induce group automorphisms of  $G^{(a)}$ , i.e.,

$$(5.18) \quad \delta_t(x \cdot y) = \delta_t(x) \cdot \delta_t(y) \quad \text{for all } x, y \in \mathbb{R}^n.$$

**Proposition 5.26.** The law group of  $G^{(a)}$  is of the form,

$$(5.19) \quad (x \cdot y)_j = x_j + y_j + \sum_{\langle \alpha \rangle + w_k = w_j} b_{kj\alpha} x^\alpha y_k + \sum_{\substack{\langle \alpha \rangle + \langle \beta \rangle = w_j \\ |\beta| \geq 2}} b_{j\alpha\beta} x^\alpha y^\beta, \quad j = 1, \dots, n,$$

where the constants  $b_{kj\alpha}$  are given by (5.9) and the  $b_{j\alpha\beta}$  are some constants.

*Proof.* The smoothness and the homogeneity (5.18) of the group law of  $G^{(a)}$  ensure us that this group law is of the form,

$$(5.20) \quad (x \cdot y)_j = \sum_{\langle \alpha \rangle + \langle \beta \rangle = w_j} b_{j\alpha\beta} x^\alpha y^\beta, \quad j = 1, \dots, n,$$

for some constants  $b_{j\alpha\beta}$ . In addition, for  $j = 1, \dots, n$ , set  $\epsilon_j = (\delta_{jk}) \in \mathbb{N}_0^n \subset \mathbb{R}^n$ . Then the equalities  $x \cdot 0 = x$  and  $0 \cdot y = y$  imply that

$$(5.21) \quad b_{j\epsilon_k 0} = b_{j0\epsilon_k} = \delta_{jk} \quad \text{for } w_k = w_j.$$

Bearing this in mind, we know from (5.9) that the vector fields  $X_j^{(a)}$  are of the form,

$$X_j^{(a)} = \partial_j + \sum_{\substack{w_k - \langle \alpha \rangle = w_j \\ \alpha \neq 0}} b_{jk\alpha} \partial_j, \quad j = 1, \dots, n,$$

for some constants  $b_{jk\alpha}$ . We also know that  $X_j^{(a)}$  is the unique left-invariant vector fields on  $G^{(a)}$  that agrees with  $\partial_j$  at  $x = 0$ . That is, for all  $f \in C^\infty(\mathbb{R}^n)$ , we have

$$X_j^{(a)} f(x) = \left. \frac{d}{dt} f(x \cdot (t\epsilon_j)) \right|_{t=0} = \sum_{1 \leq k \leq n} \left. \frac{d}{dt} (x \cdot (t\epsilon_j))_k \right|_{t=0} \partial_k f(x).$$

Using (5.20)–(5.21) we see that, as  $t \rightarrow 0^+$ , we have

$$(x \cdot (t\epsilon_j))_k = x_k + t\delta_{jk} + t \sum_{\alpha + w_j = w_k} b_{k\alpha\epsilon_j} x^\alpha + O(t^2).$$

It then follows that

$$X_j^{(a)} = \partial_j + \sum_{\alpha + w_j = w_k} b_{k\alpha\epsilon_j} x^\alpha \partial_k.$$

Comparing this to (5.9) we deduce that

$$b_{k\alpha\epsilon_j} = b_{jk\alpha} \quad \text{for } w_k - \langle \alpha \rangle = w_j.$$

Combining this with (5.20)–(5.21) shows that the product law of  $G^{(a)}$  is of the form (5.19). The proof is complete.  $\square$

## 6. MANY EXTRINSIC TANGENT GROUPS

In this section, we shall determine *all* the extrinsic tangent groups at a given point. This will show that the construction of the extrinsic tangent group in Section 6 is definitely not canonical.

Throughout this section, we continue using the notation of the previous sections. In particular,  $a$  is a point of  $M$  and  $(X_1, \dots, X_n)$  is an  $H$ -frame near  $a$ .

Let  $\mathfrak{g}^{(a)}$  and  $G^{(a)}$  be the extrinsic tangent Lie algebra and tangent group at  $a$  with respect to some privileged coordinates at  $a$  adapted to the  $H$ -frame  $(X_1, \dots, X_n)$ . Recall that the model vector fields  $X_j^{(a)}$ ,  $j = 1, \dots, n$ , provide us with a basis of  $\mathfrak{g}^{(a)}$  such that

(1) For  $j = 1, \dots, n$ , we have

$$(6.1) \quad X_j^{(a)}(0) = 0 \quad \text{and} \quad \delta_t^* X_j^{(a)} = t^{-w_j} X_j^{(a)} \quad \forall t > 0.$$

(2) For  $i, j = 1, \dots, n$ , we have

$$(6.2) \quad [X_i^{(a)}, X_j^{(a)}] = \begin{cases} \sum_{w_k = w_i + w_j} L_{ij}^k(a) X_k^{(a)} & \text{if } w_i + w_j \leq r, \\ 0 & \text{if } w_i + w_j > r. \end{cases}$$

**Definition 6.1.** We say that a Lie algebra  $\mathfrak{g} \simeq \mathfrak{g}M(a)$  is admissible when it is a Lie algebra of vector fields on  $\mathbb{R}^n$  and it admits a basis  $\{Y_j\}_{1 \leq j \leq n}$  satisfying (6.1)–(6.2).

Let  $\mathfrak{g}$  be an admissible Lie algebra in the isomorphism class of  $\mathfrak{g}M(a)$  and  $\{Y_j\}_{1 \leq j \leq n}$  a basis of  $\mathfrak{g}$  satisfying (6.1)–(6.2). Note that (6.2) means that the structure constants of  $\mathfrak{g}$  with respect to the basis  $\{Y_j\}$  agree with the structure constants of  $\mathfrak{g}M(a)$  with respect to the basis  $\{\dot{X}(a)_j\}_{1 \leq j \leq n}$ . Therefore, we see that  $\mathfrak{g}$  has a Carnot-type structure isomorphic to that  $\mathfrak{g}M(a)$  and given by the grading,

$$(6.3) \quad \mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_r, \quad \mathfrak{g}_w := \text{Span}\{Y_j; w_j = w\}.$$

We also note that, in the same way as with  $\mathfrak{g}^{(a)}$ , the homogeneity of the vector fields  $Y_j$  provided by (6.1) implies that the dilations  $\delta_t^*$ ,  $t > 0$ , acts as Lie algebra automorphisms on  $\mathfrak{g}$  and induce a grading that agrees with (6.3), so that we have

$$\mathfrak{g}_w = \{X \in \mathfrak{g}; \delta_t^* X = t^{-w} X \text{ for all } t > 0\} \quad \text{for } w = 1, \dots, r.$$

We shall realize the Lie group of  $\mathfrak{g}$  as the Carnot-type Lie group  $G$  given by  $\mathbb{R}^n$  and the exponential map,

$$(6.4) \quad \exp : \mathfrak{g} \ni X \longrightarrow \exp(X),$$

where  $\exp(X)$  is defined as in (5.15). The product of  $G$  is defined as in (5.16) by using the above exponential map. Note that any Carnot-type Lie algebra isomorphism  $\mathfrak{g} \simeq \mathfrak{g}M(a)$  gives rise to a Carnot-group isomorphism  $G \simeq GM(a)$ . Moreover, as in (5.17) we have

$$(6.5) \quad t \cdot \exp(X) = \exp(\delta_t^* X) \quad \forall X \in \mathfrak{g}.$$

Therefore, the dilations  $\delta_t$ ,  $t > 0$ , induce group isomorphisms of  $G$ . It then can be shown that the product law of  $G$  is of the form (5.16).

**Definition 6.2.** A Carnot-type group  $G \simeq GM(a)$  is called admissible when it is associated with an admissible Lie algebra  $\mathfrak{g} \simeq \mathfrak{g}M(a)$  as above.

We shall now proceed to show that any admissible Carnot-type group  $G \simeq GM(a)$  is the extrinsic tangent group with respect to some suitable privileged coordinates.

In what follows we let  $G$  be an admissible Carnot-type in the isomorphism class of  $GM(a)$ . We denote by  $\mathfrak{g}$  its Lie algebra and let  $\{Y_j\}$  be a basis of  $\mathfrak{g}$  satisfying (6.1)–(6.2). Note that, for  $j = 1, \dots, n$ , the vector field  $Y_j$  is the unique left-invariant vector field on  $G$  that agrees with  $\partial_j$  at  $x = 0$ .

**Definition 6.3.** We say map  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is *w-homogeneous* when

$$(6.6) \quad \phi(t \cdot x) = t \cdot \phi(x) \quad \text{for all } x \in \mathbb{R}^n \text{ and } t > 0.$$

*Remark 6.4.* Set  $\phi(x) = (\phi_1(x), \dots, \phi_n(x))$ . The condition (6.6) exactly means that, for all  $j = 1, \dots, n$ , the component  $\phi_j(x)$  is homogeneous of degree  $w_j$  with respect to the dilations (5.1). In particular, in view of Remark 5.3, we see that if  $\phi$  is smooth and *w-homogeneous*, then it must be a polynomial map. More precisely, each component  $\phi_j(x)$  is a linear combination of monomials  $x^\alpha$  with  $\langle \alpha \rangle = w_j$ .

**Lemma 6.5.** *There is a w-homogeneous smooth diffeomorphism  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that*

$$(6.7) \quad \phi_* X_j^{(a)} = Y_j \quad \text{for } j = 1, \dots, n.$$

*Proof.* Let  $\xi : \mathfrak{g}^{(a)} \rightarrow \mathfrak{g}$  be the linear isomorphism defined by

$$\xi(x_1 X_1^{(a)} + \dots + x_n X_n^{(a)}) = x_1 Y_1 + \dots + x_n Y_n \quad \forall x_j \in \mathbb{R}^n.$$

As the bases  $\{X_j^{(a)}\}$  and  $\{Y_j\}$  both satisfy (6.1), we see that

$$(6.8) \quad \xi(\delta_t^* X) = \delta_t^* (\xi(X)) \quad \text{for all } t > 0 \text{ and } X \in \mathfrak{g}^{(a)}.$$

Note also that the bases  $\{X_j^{(a)}\}$  and  $\{Y_j\}$  also both satisfy (6.2), and so the structure constants of  $\mathfrak{g}^{(a)}$  with respect to the basis  $\{X_j^{(a)}\}$  agree with the structure constants of  $\mathfrak{g}$  with respect to the basis  $\{Y_j\}$ . Therefore, we see that  $\xi$  is a Lie algebra isomorphism. We then define a Lie group isomorphism  $\phi : G^{(a)} \rightarrow G$  by setting

$$\phi = \exp \circ \xi \circ (\exp^{(a)})^{-1}.$$

In particular,  $\phi$  is a smooth diffeomorphism of  $\mathbb{R}^n$ . Moreover, using (5.17), (6.5) and (6.8), we see that, for all  $t > 0$ , we have

$$\phi \circ \delta_t = \exp \circ \xi \circ \delta_t^* \circ (\exp^{(a)})^{-1} = \exp \circ \delta_t^* \circ \xi \circ (\exp^{(a)})^{-1} = \delta_t \circ \phi.$$

Therefore, the diffeomorphism  $\phi$  is *w-homogeneous*.

It remains to check that  $\phi$  satisfies (6.7). Let  $j \in \{1, \dots, n\}$ . As  $\phi$  is a Lie group isomorphism the vector field  $\phi_* X_j^{(a)}$  is a left-invariant vector field on  $G$ . Furthermore, we have

$$(\exp^{(a)})'(0)(X_j^{(a)}(0)) = \partial_j, \quad \xi'(0)(X_j^{(a)}(0)) = Y_j(0), \quad \exp'(0)(Y_j(0)) = \partial_j.$$

Thus,

$$\phi_* X_j^{(a)}(0) = \phi'(0)(X_j^{(a)}(0)) = \phi'(0)(\partial_j) = \partial_j.$$

Therefore, the vector field  $\phi_* X_j^{(a)}$  is a left-invariant vector field on  $G$  and agrees with  $\partial_j$  at  $x = 0$ . As  $Y_j$  satisfy these properties, we deduce that  $\phi_* X_j^{(a)} = Y_j$ . This proves (6.7) and completes the proof.  $\square$

**Lemma 6.6.** *Let  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $w$ -homogeneous smooth diffeomorphism such that*

$$(6.9) \quad \phi_* X_j^{(a)} = X_j^{(a)} \quad \text{for } j = 1, \dots, n.$$

*Then  $\phi$  is the identity map.*

*Proof.* We note that the  $w$ -homogeneity and smoothness of  $\phi$  imply that  $\phi(0) = 0$ . Therefore, in order to prove that  $\phi = \text{id}$  we only have to show that

$$(6.10) \quad \partial_j \phi_k(x) = \delta_{jk} \quad \text{for } j, k = 1, \dots, n.$$

We also observe that (6.9) means that

$$(6.11) \quad \phi'(x)[X_j^{(a)}(x)] = X_j^{(a)}(\phi(x)) \quad \text{for } j = 1, \dots, n.$$

In particular, for  $x = 0$  we get

$$\phi'(0)[X_j^{(a)}(0)] = \phi'(0)[\partial_j] = X_j^{(a)}(0) = \partial_j.$$

This shows that  $\phi'(0) = \text{id}$ . Combining this with the  $w$ -homogeneity of  $\phi$  we deduce that  $\phi(x)$  is of the form,

$$\phi_k(x) = x_k + \sum_{\substack{\langle \alpha \rangle = w_k \\ |\alpha| \geq 2}} c_{k\alpha} x^\alpha, \quad k = 1, \dots, n,$$

for some constants  $c_{k\alpha}$ . In particular, we see that the component  $\phi_k(x)$  does not depend on the variables  $x_j$  with  $w_j \geq w_k$  and  $j \neq k$  and is linear with slope 1 with respect to  $x_k$ . Thus,

$$\partial_j \phi_k(x) = \delta_{jk} \quad \text{for } w_j \geq w_k.$$

This proves that (6.10) holds for  $w_k - w_j \leq 0$ .

As (5.9) shows, the vector fields  $X_j^{(a)}$ ,  $j = 1, \dots, n$ , are of the form,

$$X_j^{(a)} = \partial_j + \sum_{w_k > w_j} b_{kj}(x) \partial_k,$$

where  $b_{jk}(x)$  is a linear combination of monomials  $x^\alpha$  with  $\langle \alpha \rangle = w_k - w_j$ . In particular, we see that  $b_{jk}(x)$  does not depend on the variables  $x_l$  with  $w_l > w_k - w_j$ . In addition, in terms of the coefficients  $b_{jk}(x)$  the equalities (6.11) imply that

$$(6.12) \quad \partial_j \phi_k(x) + \sum_{w_j < w_l \leq k} b_{jl}(x) \partial_l \phi_k(x) = b_{jk}(\phi(x)) \quad \text{for } w_k > w_j.$$

We shall now proceed to prove (6.10) by induction on  $w_k - w_j$ . We already know that (6.10) holds when  $w_k - w_j < 0$ . Assume that (6.10) holds for  $w_k - w_j < m$  for some  $m \in \mathbb{N}$ . We remark that if  $w_k \leq m$ , then, for all  $j = 1, \dots, n$ , we have  $w_k - w_j \leq m - 1 < m$ , and so  $\partial_j \phi_k(x) = \delta_{jk}$ . As  $\phi_k(0) = 0$ , we then deduce that

$$(6.13) \quad \phi_k(x) = x_k \quad \text{for } w_k \leq m.$$

Let  $j$  and  $k$  be positive integers  $\leq n$  such that  $w_j < w_k \leq w_j + m$ . Then (6.12) gives

$$\partial_k \phi_j(x) = b_{jk}(\phi(x)) - \sum_{w_j < w_l \leq k} b_{jl}(x) \partial_l \phi_k(x).$$

If  $w_j < w_l \leq w_k$ , then  $w_k - w_l < w_k - w_j \leq m$ , and so  $\partial_l \phi_k(x) = \delta_{lk}$ . Moreover, as mentioned above, the coefficient  $b_{jk}(x)$  depends only on the variables  $x_p$  with  $w_p \leq w_k - w_j \leq m$ . As (6.13) ensures us that  $\phi_p(x) = x_p$  for  $w_p \leq m$ , we deduce that  $b_{jk}(\phi(x)) = b_{jk}(x)$ . It then follows that

$$\partial_k \phi_j(x) = b_{jk}(\phi(x)) - \sum_{w_j < w_l \leq k} b_{jl}(x) \delta_{lk} = b_{jk}(x) - b_{jk}(x) = 0.$$

This shows that if (6.10) holds for  $w_k - w_j < m$ , then it holds for  $w_k - w_j \leq m$ . As it holds for  $w_k - w_j < 0$ , we deduce that it holds for all  $j, k = 1, \dots, n$ . As mentioned above this shows that  $\phi$  is the identity map. The proof is complete.  $\square$

We shall now combine Lemma 6.5 and Lemma 6.6 to get the following result.



**Proposition 6.7.** *There is a unique  $w$ -homogeneous smooth diffeomorphism  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that*

$$(6.14) \quad \phi_* X_j^{(a)} = Y_j \quad \text{for } j = 1, \dots, n.$$

*Proof.* The existence of  $\phi$  is ensured by Lemma 6.5. Suppose that  $\psi$  is another such diffeomorphism. Then the transition map  $\phi^{-1} \circ \psi$  is a  $w$ -homogeneous smooth diffeomorphism of  $\mathbb{R}^n$ . Moreover, for  $j = 1, \dots, n$ , we have

$$(\phi^{-1} \circ \psi)_* X_j^{(a)} = \phi^*(\psi_* X_j^{(a)}) = \phi^* Y_j = X_j^{(a)}.$$

It then follows from Lemma 6.6 that  $\phi^{-1} \circ \psi = \text{id}$ , i.e.,  $\psi = \phi$ . Therefore, we see that  $\phi$  is the unique  $w$ -homogeneous smooth diffeomorphism of  $\mathbb{R}^n$  satisfying (6.14). The proof is complete.  $\square$

*Remark 6.8.* The  $w$ -homogeneity and the smoothness of  $\phi(x)$  imply that it is of the form,

$$\phi_k(x) = \sum_{\langle \alpha \rangle = w_k} c_{k\alpha} x^\alpha, \quad k = 1, \dots, n,$$

for some constants  $c_{k\alpha}$ . In addition, using (6.14) we get

$$\phi'(0)[\partial_j] = \phi'(0)[X_j^{(a)}(0)] = Y_j(0) = \partial_j.$$

Therefore, we see that  $\phi'(0) = \text{id}$ , and hence  $\phi(x)$  is of the form,

$$(6.15) \quad \phi_k(x) = x_k + \sum_{\substack{\langle \alpha \rangle = w_k \\ |\alpha| \geq 2}} c_{k\alpha} x^\alpha, \quad k = 1, \dots, n.$$

We are now in a position to prove that the main result of this section.

**Theorem 6.9.** *Let  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be the  $w$ -homogeneous smooth diffeomorphism provided by Proposition 6.7. Then the change of variable  $x \rightarrow \phi(x)$  provides us with privileged coordinates at  $a$  adapted to  $(X_1, \dots, X_n)$  with respect to which  $\mathfrak{g}^{(a)} = \mathfrak{g}$  and  $G^{(a)} = G$ .*

*Proof.* We note that (6.15) implies that  $\phi(0) = 0$  and  $\phi'(0) = \text{id}$ , and so, for  $j = 1, \dots, n$ , we have

$$\phi_* X_j(0) = \phi'(0)[X_j(0)] = X_j(0) = \partial_j.$$

Therefore, we see that the local coordinates  $x' = \phi(x)$  are linearly adapted at  $a$  to the  $H$ -frame  $(X_1, \dots, X_n)$ . In addition, let us denote by  $U$  the range of the local coordinates  $(x_1, \dots, x_n)$ , and set  $U' = \phi(U)$ . Using the  $w$ -homogeneity of  $\phi$  we see that, as  $t \rightarrow 0^+$  and in  $\mathcal{X}(U')$ , we have

$$t^{w_j} \delta_t^* [\phi_* X_j] = \phi_* [t^{w_j} \delta_t^* X_j] = \phi_* X_j^{(a)} + O(t) = Y_j + O(t).$$

Using Proposition 5.20 we then deduce that the local coordinates  $x' = \phi(x)$  are privileged coordinates at  $a$  adapted to  $(X_1, \dots, X_n)$ . Moreover, the associated extrinsic tangent Lie algebra is spanned by the vector fields  $Y_j$ ,  $j = 1, \dots, n$ , and hence agrees with  $\mathfrak{g}$ . It then follows that the extrinsic tangent group in these new privileged coordinates is precisely  $G$ . The proof is complete.  $\square$

*Example 6.10.* It is worth illustrating Theorem 6.9 by looking at the special case of a contact structure  $(M^{2n+1}, H)$ . In this case, the tangent group bundle  $GM$  is a fiber bundle with fibers isomorphic to the Heisenberg group  $\mathbb{H}^{2n+1}$ , so that we may assume that  $GM(a) = \mathbb{H}^{2n+1}$ . We define  $\mathbb{H}^{2n+1}$  as  $\mathbb{R}^{n+1}$  with the group law (3.3). A basis of its Lie algebra of left-invariant vector fields is then given by

$$X_0 = \partial_{x_0}, \quad X_j = \partial_{x_j} - x_{n+j} \partial_{x_0}, \quad X_j = \partial_{x_{n+j}} + x_{n+j} \partial_{x_0}, \quad j = 1, \dots, n.$$

These vector fields are homogeneous with respect to the dilations (3.4). Let  $b = (b_{jk})$  be a real symmetric  $2n \times 2n$ -matrix, and denote by  $\mathfrak{h}$  the subspace of  $T\mathbb{R}^{2n+1}$  spanned by the vector fields,

$$Y_0 = \partial_{x_0}, \quad Y_j = X_j + \sum_{1 \leq k \leq 2n} b_{jk} x_k \partial_{x_0}, \quad j = 1, \dots, n.$$

We note that  $Y_j$  agrees with  $X_j(0) = \partial_j$  at  $x = 0$  and is homogeneous with respect to the dilations (3.4). We also observe that  $Y_0$  commutes with the other vector fields  $Y_j$ . Moreover, as the matrix  $b$  is symmetric, for  $j, k = 1, \dots, 2n$ , we have

$$\begin{aligned} [Y_j, Y_k] &= [X_j, X_k] + [\partial_{x_j}, b_{kj}x_j\partial_{x_0}] + [b_{jk}x_k\partial_{x_0}, \partial_{x_k}] \\ &= [X_j, X_k] + b_{jk}([\partial_{x_j}, x_j] + [x_k, \partial_{x_k}])\partial_{x_0} \\ &= [X_j, X_k] \end{aligned}$$

It then follows that  $\mathfrak{h}$  is a Lie algebra isomorphic to the Heisenberg Lie algebra  $\mathfrak{h}^{2n+1} = \mathfrak{g}M(a)$  and the basis  $\{Y_j\}$  satisfy (6.1)–(6.2), so that  $\mathfrak{h}$  is an admissible Lie algebra. This allows us to obtain a whole family of extrinsic tangent groups  $G^{(a)}$  parametrized by the space of real symmetric  $2n \times 2n$  matrices. This provides us with a huge class of extrinsic tangent groups.

## 7. CARNOT COORDINATES

In this section, we shall refine the construction of the privileged coordinates of Section 6 to get a system of privileged coordinates in which the extrinsic tangent group agrees with the intrinsic tangent group constructed in Section 2. We shall call these coordinates *Carnot coordinates*. This will remedy the lack of canonicalness of the extrinsic tangent group exhibited by Theorem 6.9.

Throughout this section, we continue using the notation of the previous sections. In particular,  $a$  is a point of  $M$  and  $(X_1, \dots, X_n)$  is an  $H$ -frame near  $a$ .

**7.1. Construction of Carnot coordinates.** Recall that the  $H$ -frame  $(X_1, \dots, X_n)$  yields a basis  $\{\dot{X}_j(a)\}$  of  $\mathfrak{g}M(a)$ , where  $\dot{X}_j(a)$  is the class of  $X_j(a)$  in  $\mathfrak{g}_{w_j}M(a) = H_{w_j}(a)/H_{w_j-1}(a)$ . This then provides us with a global system of coordinates on  $\mathfrak{g}M(a)$  and  $GM(a)$ , using which we can identify  $GM(a)$  as a Carnot-type group whose underlying manifold is  $\mathbb{R}^n$ . Incidentally, we can regard the vector fields  $X_j^a$  as vector fields on  $\mathbb{R}^n$ . They form a basis of the Lie algebra of left-invariant vector fields on  $GM(a)$  satisfying (6.1)–(6.2). Therefore, under these identifications,  $GM(a)$  is an admissible Carnot-type group in the sense of Definition 6.2. This leads us to the following definition.

**Definition 7.1.** We say that privileged coordinates  $(x_1, \dots, x_n)$  at  $a$  adapted to the  $H$ -frame  $(X_1, \dots, X_n)$  are Carnot coordinates when, under the above identification, the extrinsic tangent group  $G^{(a)}$  agrees with  $GM(a)$ .

The following result provides us with a useful and simple characterization of Carnot coordinates.

**Proposition 7.2.** Let  $(x_1, \dots, x_n)$  be local coordinates that are linearly adapted at  $a$  to the  $H$ -frame  $(X_1, \dots, X_n)$ , and denote by  $U$  their range. Then the following are equivalent:

- (1) The local coordinates  $(x_1, \dots, x_n)$  are Carnot coordinates at  $a$  adapted to the  $H$ -frame  $(X_1, \dots, X_n)$ .
- (2) For  $j = 1, \dots, n$  and as  $t \rightarrow 0^+$ , we have

$$(7.1) \quad t^{w_j} \delta_t^* X_j = X_j^a + O(t) \quad \text{in } \mathcal{X}(U).$$

*Proof.* Assume that  $(x_1, \dots, x_n)$  are Carnot coordinates. Then, for  $j = 1, \dots, n$ , both  $X_j^{(a)}$  and  $X_j^a$  are left-invariant vector fields on  $G^{(a)} = GM(a)$  that agree with  $\partial_j$  at  $x = 0$ . Therefore, they agree everywhere and we obtain (7.1). Conversely, suppose that, for  $j = 1, \dots, n$  and as  $t \rightarrow 0^+$ , we have

$$t^{w_j} \delta_t^* X_j = X_j^a + O(t) \quad \text{in } \mathcal{X}(U).$$

This means that, for  $j = 1, \dots, n$ , in these coordinates the vector field  $X_j$  has weight  $-w_j$  and model vector field  $X_j^a$ . Using Proposition 5.20 we then deduce that  $(x_1, \dots, x_n)$  are privileged coordinates. We also see that  $G^{(a)}$  and  $GM(a)$  are Lie groups with underlying space  $\mathbb{R}^n$  whose Lie algebras of left-invariant vector field are spanned by the vector fields  $X_j^a$ . Therefore, these two groups agree, and hence  $(x_1, \dots, x_n)$  are Carnot coordinates. This completes the proof.  $\square$

In what follows, we let  $(x_1, \dots, x_n)$  be privileged coordinates at  $a$  adapted to the  $H$ -frame  $(X_1, \dots, X_n)$  and denote by  $U$  their range. In these coordinates we write

$$X_j = \sum_{1 \leq k \leq n} b_{jk}(x) \partial_{x_k}, \quad j = 1, \dots, n,$$

where the coefficients  $b_{jk}(x)$  are smooth. In addition, for  $j = 1, \dots, n$ , we denote by  $X_j^{(a)}$  the model vector field of  $X_j$ . We then let  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be the unique  $w$ -homogeneous smooth diffeomorphism of  $\mathbb{R}^n$  such that  $\phi_* X_j = X_j^{(a)}$  for  $j = 1, \dots, n$ . As it turns out, the map  $\phi$  has a very simple expression.

**Lemma 7.3.** *For all  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ , we have*

$$\phi^{-1}(x) = \exp \left( x_1 X_1^{(a)} + \dots + x_n X_n^{(a)} \right).$$

*Proof.* Let  $\mathfrak{g}^a$  be the Lie algebra of left-invariant vector fields on  $GM(a)$  and denote by  $\exp$  its exponential map (6.4). Then it follows from the proof of Lemma 6.5 that

$$\phi = \exp \circ \xi \circ (\exp^{(a)})^{-1},$$

where  $\exp^{(a)}$  is the exponential map (5.15) and  $\xi$  is the Lie algebra isomorphism,

$$\xi : \mathfrak{g}^{(a)} \ni x_1 X_1^{(a)} + \dots + x_n X_n^{(a)} \longrightarrow x_1 X_1^a + \dots + x_n X_n^a \in \mathfrak{g}^a.$$

Moreover, by functoriality the exponential map of  $\mathfrak{g}^a$  is the composition of the exponential map of  $GM(a)$  with the Lie algebra isomorphism,

$$\mathfrak{g}^a \ni x_1 X_1^a + \dots + x_n X_n^a \longrightarrow (x_1, \dots, x_n) \in GM(a).$$

As the exponential map of  $GM(a)$  is the identity map, we deduce that the above map agrees with the exponential map of  $\mathfrak{g}^a$ . We then deduce that, for all  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ , we have

$$\phi^{-1}(x) = \exp^{(a)} \circ \xi^{-1} (x_1 X_1^a + \dots + x_n X_n^a) = \exp \left( x_1 X_1^{(a)} + \dots + x_n X_n^{(a)} \right).$$

The lemma is thus proved.  $\square$

The following result shows how to modify the coordinates  $(x_1, \dots, x_n)$  to get Carnot coordinates.

**Theorem 7.4.** *The change of variable  $x \rightarrow \phi(x)$  is the unique  $w$ -homogeneous change of variable that provides us with Carnot coordinates at  $a$  adapted to  $(X_1, \dots, X_n)$ .*

*Proof.* It immediately follows from Theorem 6.9 that the change of variables  $x \rightarrow \phi(x)$  provides us with Carnot coordinates at  $a$  adapted to  $(X_1, \dots, X_n)$ . Suppose that  $x \rightarrow \tilde{\phi}(x)$  is another  $w$ -homogeneous change of variables that provides us with Carnot coordinates at  $a$  adapted to  $(X_1, \dots, X_n)$ . Let us denote by  $U$  the range of the coordinates  $(x_1, \dots, x_n)$  and set  $\tilde{U} = \tilde{\phi}(U)$ . Let  $j \in \{1, \dots, n\}$ . Using the  $w$ -homogeneity of  $\tilde{\phi}$  and Proposition 7.2, we see that, as  $t \rightarrow 0^+$  and in  $\mathcal{X}(\tilde{U})$ , we have

$$\tilde{\phi}_* X_j^{(a)} = \lim_{t \rightarrow 0^+} \tilde{\phi}_* [t^{w_j} \delta_t^* X_j] = \lim_{t \rightarrow 0^+} t^{w_j} \delta_t^* [\tilde{\phi}_* X_j] = X_j^a.$$

The uniqueness contents of Proposition 6.7 then imply that  $\tilde{\phi} = \phi$ . Therefore, we see that  $x \rightarrow \phi(x)$  is the unique  $w$ -homogeneous change of variables that provides us with Carnot coordinates at  $a$  adapted to  $(X_1, \dots, X_n)$ . The proof is complete.  $\square$

As it turns out, the diffeomorphism  $\phi(x) = (\phi_1(x), \dots, \phi_n(x))$  can be determined effectively from the coefficients  $b_{jk}(x)$ . Namely, we have the following statement.

**Proposition 7.5.** *For  $j = 1, \dots, n$ , we have*

$$(7.2) \quad \phi_j(x) = x_j + \sum_{\substack{\langle \alpha \rangle = w_j \\ |\alpha| \geq 2}} c_{j\alpha} x^\alpha, \quad j = 1, \dots, n,$$

where  $c_{j\alpha}$  is a universal polynomial in the partial derivatives  $\partial^\beta b_{kl}(0)$  with  $w_k + \langle \beta \rangle = w_l \leq w_j$  and  $\beta \neq 0$ .

*Proof.* Let  $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$  and set  $X = \xi_1 X_1^{(a)} + \dots + \xi_n X_n^{(a)}$ . In addition, for  $t \geq 0$  set  $x(t) = \exp(tX)(0) = (x_1(t), \dots, x_n(t))$ , i.e.,  $x(t)$  is the solution of the initial problem,

$$\dot{x}(t) = X(x(t)), \quad x(0) = 0.$$

In view of (5.9) we have

$$X_j^{(a)} = \partial_j + \sum_{\substack{w_k - \langle \alpha \rangle = w_j \\ \alpha \neq 0}} b_{jk\alpha}^{(a)} x^\alpha \partial_k, \quad \text{where } b_{jk\alpha}^{(a)} = \frac{1}{\alpha!} \partial^\alpha b_{jk}(0).$$

Thus,

$$X = \sum_{1 \leq j \leq n} \xi_j \partial_j + \sum_{\substack{w_k - \langle \alpha \rangle = w_j \\ \alpha \neq 0}} b_{jk\alpha}^{(a)} \xi_j x^\alpha \partial_k = \sum_{j=1}^n \left( \xi_j + \sum_{\substack{w_k + \langle \alpha \rangle = w_j \\ \alpha \neq 0}} b_{kj\alpha}^{(a)} \xi_k x^\alpha \right) \partial_j.$$

We then see that  $(x_1(t), \dots, x_n(t))$  is solution of the ODE system,

$$(7.3) \quad \dot{x}_j(t) = \xi_j + \sum_{\substack{w_k + \langle \alpha \rangle = w_j \\ \alpha \neq 0}} b_{kj\alpha}^{(a)} \xi_k x(t)^\alpha, \quad j = 1, \dots, n.$$

We observe that if  $w_k + \langle \alpha \rangle = w_j$  and  $|\alpha| \geq 2$ , then  $w_k < w_j$  and  $x(t)^\alpha$  may only involve the  $x_l(t)$  with  $w_l < w_j - w_k$ . Therefore, using the initial condition  $x(0) = 0$ , the ODE system (7.3) is triangular, and hence can be solved recursively. We obtain

$$\begin{aligned} x_j(t) &= t\xi_j && \text{when } w_j = 1, \\ x_j(t) &= t\xi_j + \frac{1}{2}t^2 \sum_{\substack{w_k + \langle \alpha \rangle = w_j \\ |\alpha| \geq 2}} b_{kj\alpha}^{(a)} \xi_k \xi^\alpha && \text{when } w_j = 2. \end{aligned}$$

More generally, an induction on  $w_j$  shows that

$$(7.4) \quad x_j(t) = t\xi_j + \sum_{\substack{\langle \alpha \rangle = w_j \\ |\alpha| \geq 2}} \hat{c}_{j\alpha} t^{|\alpha|} \xi^\alpha,$$

where  $\hat{c}_{j\alpha}$  is a universal polynomial in the partial derivatives  $\partial^\beta b_{kl}(0) = \beta! b_{jk\beta}^{(a)}$  with  $w_k + \langle \alpha \rangle = w_l \leq w_j$  and  $\beta \neq 0$ .

Bearing this in mind, let  $x = (x_1, \dots, x_n)$  and take  $\xi = \phi(x)$ . By Lemma 7.3 we have

$$x = \phi^{-1}(\xi) = \exp(X) = \exp(tX)(0)|_{t=1} = x(1).$$

Therefore, using (7.4) we obtain

$$x_j = x_j(1) = \phi_j(x) + \sum_{\substack{\langle \alpha \rangle = w_j \\ |\alpha| \geq 2}} \hat{c}_{j\alpha} \phi(x)^\alpha, \quad j = 1, \dots, n.$$

Thus,

$$\phi_j(x) = x_j - \sum_{\substack{\langle \alpha \rangle = w_j \\ |\alpha| \geq 2}} \hat{c}_{j\alpha} \phi(x)^\alpha, \quad j = 1, \dots, n.$$

This shows that  $\phi_j(x)$  is recursively determined from the  $\phi_k(x)$  with  $w_k < w_j$ . An induction on  $w_j$  then shows that

$$\hat{x}_j = x_j + \sum_{\substack{\langle \alpha \rangle = w_j \\ |\alpha| \geq 2}} c_{j\alpha} x^\alpha,$$

where  $c_{j\alpha}$  is a universal polynomial in the  $\hat{c}_{k\beta}$  with  $\langle \beta \rangle = w_k \leq w_j$ , and hence is a universal polynomial in the partial derivatives  $\partial^\beta b_{kl}(0)$  with  $w_k + \langle \beta \rangle = w_l \leq w_j$  and  $\beta \neq 0$ . This proves the result.  $\square$

*Remark 7.6.* We refer to [JvE] for an alternative construction of Carnot coordinates in the special case of step 2 Carnot manifolds.

**7.2. Normal Carnot coordinates.** Combining Theorem 7.4 with the construction of privileged coordinates explained in Section 4 enables us to obtain an *effective* construction of Carnot coordinates as follows.

Let  $(x_1, \dots, x_n)$  be local coordinates near  $a$  with range  $U$ . In addition, let  $T_a$  be the affine map from Lemma 4.8 and  $\psi_a(x)$  the polynomial diffeomorphism given Definition 4.18. Recall that by Proposition 4.19 the change of variables  $x \rightarrow \psi_a \circ T_a(x)$  provides us with privileged coordinates at  $a$  adapted to  $(X_1, \dots, X_n)$ . For  $j = 1, \dots, n$ , we denote by  $X_j^{(a)}$  the model vector field of  $X_j$  in these privileged coordinates.

**Definition 7.7.**

- (1) The map  $\phi_a : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the unique  $w$ -homogeneous smooth diffeomorphism such that  $(\phi_a)_* X_j^{(a)} = X_j^a$  for  $j = 1, \dots, n$ .
- (2) The map  $\varepsilon_a : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is given by the composition

$$\varepsilon_a = \phi_a \circ \psi_a \circ T_a.$$

Using Theorem 7.4 we immediately obtain the following result.

**Theorem 7.8.** *The change of variables  $x \rightarrow \varepsilon_a(x)$  provides us with Carnot coordinates at  $a$  adapted to the  $H$ -frame  $(X_1, \dots, X_n)$ .*

**Definition 7.9.** The local coordinates provided by the change of variables  $x \rightarrow \varepsilon_a(x)$  are called normal Carnot coordinates at  $a$  adapted to the  $H$ -frame  $(X_1, \dots, X_n)$ . The map  $x \rightarrow \varepsilon_a(x)$  is called normal Carnot coordinate map.

*Remark 7.10.* For Heisenberg manifolds the normal Carnot coordinates agree with the anti-symmetric  $y$ -coordinates of [BG], where they were used in the explicit construction of the principal symbol of a sub-Laplacian. These coordinates are also called Heisenberg coordinates in [Po1]. It was observed there that in these coordinates the intrinsic and extrinsic tangent groups actually agree.

*Remark 7.11.* We refer to Proposition 9.3 for an explicit computation of normal Carnot coordinates for Carnot-type groups.

*Remark 7.12.* Set  $\hat{\varepsilon}_a = \phi_a \circ \psi_a$ . Then  $\varepsilon_a(x) = \hat{\varepsilon}_a \circ T_a$ . Moreover, it follows from (4.10) and (7.2) that  $\hat{\varepsilon}_a(x)$  is of the form,

$$(\hat{\varepsilon}_a)_j(x) = x_j + \sum_{\substack{\langle \alpha \rangle \leq w_j \\ |\alpha| \geq 2}} c_{j,\alpha} x^\alpha, \quad j = 1, \dots, n,$$

where the constants  $c_{j,\alpha}$  are as follows:

- For  $\langle \alpha \rangle = w_j$  they agree with the constants  $c_{j,\alpha}$  in (7.2).
- For  $\langle \alpha \rangle < w_j$  they are universal polynomials in the constants  $c_{k,\beta}$  and  $a_{l,\gamma}$ , with  $\langle \beta \rangle = w_k < w_j$  and  $\langle \gamma \rangle < w_l < w_j$ , where the constants  $a_{l,\gamma}$  are given by (4.10).

We refer to Proposition 10.1 for a more precise description of the constants  $c_{j,\alpha}$ .

## 8. WEIGHTED APPROXIMATION OF MULTI-VALUED MAPS

In this section, we extend to multi-valued maps the weighted approximation of Section 5. In what follows, we let  $(w_1, \dots, w_n)$  and  $(w_1, \dots, w_{n'})$  be weight sequences taking values in  $\{1, \dots, r\}$ . When  $n' = n$  we shall tacitly assume that the two weight sequences agree. In addition, we will use the same notation for both families of dilations (5.1) associated with these weight sequences. Note that the dilations associated with the former weight sequence are isomorphisms of  $\mathbb{R}^n$ , while the latter are isomorphisms of  $\mathbb{R}^{n'}$ .

In the rest of the paper we will make great use of the following lemma.

**Lemma 8.1.** *Let  $\Theta : \mathcal{U} \rightarrow \mathbb{R}^{n'}$  be a smooth map, where  $\mathcal{U}$  is an open subset of  $\mathbb{R}^p \times [0, \infty)$  for some  $p \geq 1$ . In addition, let  $m \in \mathbb{N}$  and set  $\partial \mathcal{U} = \mathcal{U} \cap (\mathbb{R}^p \times \{0\})$ . Then the following are equivalent:*

(i) For every point  $(x, 0) \in \partial\mathcal{U}$ , we have

$$(8.1) \quad t^{-1} \cdot \Theta(x, t) = O(t^m) \quad \text{as } t \rightarrow 0^+.$$

(ii) There is a unique smooth map  $\tilde{\Theta} : \mathcal{U} \rightarrow \mathbb{R}^{n'}$  such that

$$t^{-1} \cdot \Theta(x, t) = t^m \tilde{\Theta}(x, t) \quad \text{for all } (x, t) \in \mathcal{U}.$$

*Proof.* The proof is an elementary application of Taylor's formula with integral remainder, but we shall give the details for sake of completeness and reader's convenience. It is immediate that (ii) implies (i). Moreover, if there is a smooth map  $\tilde{\Theta} : \mathcal{U} \rightarrow \mathbb{R}^{n'}$  satisfying (8.1), then it must be unique and it agrees with  $t^{-m} [t^{-1} \cdot \Theta(x, t)]$  on  $\mathcal{U}_0 := \{(x, t) \in \mathcal{U}; t > 0\}$ . Therefore, we only have to prove that if (i) is satisfied, then  $t^{-m} [t^{-1} \cdot \Theta(x, t)]$  has a smooth extension up to  $\partial\mathcal{U}$ .

Assume that (i) holds and let  $(x_0, 0) \in \partial\mathcal{U}$ . As  $\mathcal{U}$  is open subset of  $\mathbb{R}^p \times [0, \infty)$ , we can find an open neighborhood  $V$  of  $x_0$  in  $\mathbb{R}^p$  and  $t_0 > 0$  such that  $V \times [0, t_0] \subset \mathcal{U}$ . For  $j = 1, \dots, n'$ , let us denote by  $\Theta_j(x, t)$  the  $j$ -th component of  $\Theta(x, t)$ , so that  $\Theta(x, t) = (\Theta_1(x, t), \dots, \Theta_{n'}(x, t))$ . Note also that (8.1) implies that, for all  $x \in V$  and for  $j = 1, \dots, n'$ , we have

$$(8.2) \quad \Theta_j(x, t) = O(t^{w'_j+m}) \quad \text{as } t \rightarrow 0^+.$$

Let  $j \in \{1, \dots, n'\}$  and set  $m_j = m + w'_j$ . By Taylor's formula, for all  $(x, t) \in V \times [0, t_0]$ , we have

$$\Theta_j(x, t) = \sum_{\ell < m_j} \frac{1}{\ell!} \partial_t^\ell \Theta_j(x, 0) t^\ell + t^{m_j} \tilde{\Theta}_j(x, t),$$

where we have set  $\tilde{\Theta}_j(x, t) := \frac{1}{(m_j-1)!} \int_0^1 (1-s)^{m_j-1} \partial_t^{m_j} \Theta_j(x, ts) ds$ . It is immediate that  $\tilde{\Theta}_j(x, t)$  is a smooth function on  $V \times [0, t_0]$ . Moreover, the asymptotics (8.2) imply that  $\partial_t^\ell \Theta_j(x, 0) = 0$  for  $\ell < m_j$ , and hence  $\Theta_j(x, t) = t^{m_j} \tilde{\Theta}_j(x, t)$ . Therefore, for all  $(x, t) \in V \times [0, t_0]$ , we have

$$t^{-m} [t^{-1} \cdot \Theta(x, t)] = (t^{-m_1} \Theta_1(x, t), \dots, t^{-m_{n'}} \Theta_{n'}(x, t)) = (\tilde{\Theta}_1(x, t), \dots, \tilde{\Theta}_{n'}(x, t)).$$

It then follows that  $t^{-m} [t^{-1} \cdot \Theta(x, t)]$  has a smooth extension to  $V \times [0, t_0]$ . It thus extends smoothly up to  $\partial\mathcal{U}$ . The proof is complete.  $\square$

**Definition 8.2.** Let  $m \in \mathbb{N}_0$  and  $\Theta(x) = (\Theta_1(x), \dots, \Theta_{n'}(x))$  a smooth map from an open neighborhood of the origin  $0 \in \mathbb{R}^n$  to  $\mathbb{R}^{n'}$ . We shall say that  $\Theta(x)$  is  $O_w(\|x\|^{w+m})$ , and write  $\Theta(x) = O_w(\|x\|^{w+m})$ , when, near  $x = 0$ , we have

$$\Theta_j(x) = O(\|x\|^{w_j+m}) \quad \text{for } j = 1, \dots, n'.$$

We have the following characterizations of being  $O_w(\|x\|^{w+m})$ .

**Lemma 8.3.** Let  $\Theta(x) = (\Theta_1(x), \dots, \Theta_{n'}(x))$  be a smooth map from  $U$  to  $\mathbb{R}^{n'}$ , where  $U$  is an open neighborhood of the origin  $0 \in \mathbb{R}^n$ . In addition, set  $\mathcal{U} = \{(x, t) \in U \times [0, \infty); t \cdot x \in U\}$ . Then the following are equivalent:

- (1) The function  $\Theta(x)$  is  $O_w(\|x\|^{w+m})$  near  $x = 0$ .
- (2) For  $j = 1, \dots, n'$ , the component  $\Theta_j(x)$  has weight  $\geq w'_j + m$ .
- (3) For all  $x \in \mathbb{R}^n$ , we have

$$t^{-1} \cdot \Theta(t \cdot x) = O(t^m) \quad \text{as } t \rightarrow 0^+.$$

- (4) As  $t \rightarrow 0^+$ , we have

$$t^{-1} \cdot \Theta(t \cdot x) = O(t^m) \quad \text{in } C^\infty(U, \mathbb{R}^{n'}).$$

- (5) There is a smooth function on  $\tilde{\Theta}(x, t)$  from  $\mathcal{U}$  to  $\mathbb{R}^{n'}$  such that

$$(8.3) \quad t^{-1} \cdot \Theta(t \cdot x) = t^m \tilde{\Theta}(x, t) \quad \text{for all } (x, t) \in \mathcal{U}.$$

*Proof.* Let us first establish the equivalence of (3), (4) and (5). It is immediate that (4) implies (3). Moreover, the equivalence of (3) and (5) is an immediate consequence of Lemma 8.1. In addition, if (5) holds, then the smoothness of  $\tilde{\Theta}(x, t)$  implies that  $\tilde{\Theta}(x, t)$  is  $O(1)$  in  $C^\infty(U, \mathbb{R}^{n'})$  as  $t \rightarrow 0^+$ , and hence  $t^{-1} \cdot \Theta(t \cdot x) = t^m \tilde{\Theta}(x, t)$  is  $O(t^m)$  in  $C^\infty(U, \mathbb{R}^{n'})$ . That is, (4) holds. Therefore the properties (3), (4) and (5) are equivalent.

Assume that  $\Theta(x) = O_w(\|x\|^{w+m})$  near  $x = 0$ . This means that  $\|x\|^{-1} \cdot \Theta(x) = O(\|x\|^m)$  near  $x = 0$ . Therefore, for all  $x \in \mathbb{R}^n \setminus 0$  and as  $t \rightarrow 0^+$ , we have

$$t^{-1} \cdot \Theta(t \cdot x) = \|x\| \cdot (\|t \cdot x\| \cdot \Theta(t \cdot x)) = O(\|t \cdot x\|) = O(t^m).$$

This shows that (1) implies (3).

Let  $j \in \{1, \dots, n\}$ . By Lemma 5.4 there are functions  $R_{j\alpha} \in C^\infty(U)$ ,  $|\alpha| \leq w'_j + m \leq \langle \alpha \rangle$ , such that

$$\Theta_j(x) = \sum_{\langle \alpha \rangle < w'_j + m} \frac{1}{\alpha!} x^\alpha \partial_x^\alpha \Theta_j(0) + \sum_{|\alpha| \leq w'_j + m \leq \langle \alpha \rangle} x^\alpha R_{j\alpha}(x).$$

Therefore, for all  $t > 0$ , on  $\delta_{t^{-1}}(U)$  we have

$$\Theta_j(t \cdot x) = \sum_{\langle \alpha \rangle < w'_j + m} \frac{1}{\alpha!} t^\alpha x^\alpha \partial_x^\alpha \Theta_j(0) + \sum_{|\alpha| \leq w'_j + m \leq \langle \alpha \rangle} t^{\langle \alpha \rangle} x^\alpha R_{j\alpha}(t \cdot x).$$

We then see that  $\Theta_j(t \cdot x) = O(t^{w'_j + m})$  if and only if  $\partial_x^\alpha \Theta_j(0) = 0$  for  $\langle \alpha \rangle < w'_j + m$ , i.e.,  $\Theta_j(x)$  has weight  $\geq w'_j + m$ . It then follows that (2) and (3) are equivalent.

In order to complete the proof it is enough to show that (5) implies (1). Assume that (5) is satisfied and let  $x \in U \setminus 0$ . Substituting  $\|x\|$  for  $t$  and  $\|x\|^{-1} \cdot x$  for  $x$  in (8.3) gives

$$\|x\|^{-1} \cdot \Theta(x) = \|x\|^m \tilde{\Theta}(\|x\|^{-1}x, \|x\|).$$

The smoothness of  $\tilde{\Theta}$  on  $\mathcal{U}$  implies that  $\tilde{\Theta}(\|x\|^{-1}x, \|x\|) = O(1)$  near  $x = 0$ . It then follows that  $\|x\|^{-1} \cdot \Theta(x) = O(\|x\|^m)$  near  $x = 0$ , i.e.,  $\Theta(x) = O_w(\|x\|^{w+m})$ . This shows that (5) implies (1). The proof is complete.  $\square$

**Lemma 8.4.** *Let  $\phi : U_1 \rightarrow U_2$  be a smooth diffeomorphism, where  $U_1$  and  $U_2$  are open neighborhoods of the origin  $0 \in \mathbb{R}^n$ . Assume further that  $\phi(0) = 0$  and, near  $x = 0$ , we have*

$$(8.4) \quad \phi(x) = \hat{\phi}(x) + O_w(\|x\|^{w+m}),$$

*where  $\hat{\phi}(x)$  is  $w$ -homogeneous smooth map. Then  $\hat{\phi}(x)$  is a diffeomorphism and, near  $x = 0$ , we have*

$$\phi^{-1}(x) = \hat{\phi}^{-1}(x) + O_w(\|x\|^{w+m}).$$

*Proof.* We observe that (8.4) implies that  $\phi(x)$  and  $\hat{\phi}(x)$  has same differential at the origin. It then follows that  $\hat{\phi}(x)$  is a diffeomorphism near the origin. Its  $w$ -homogeneity then implies it is a diffeomorphism of  $\mathbb{R}^n$ .

For  $l = 1, 2$ , set  $\mathcal{U}_l = \{(x, t) \in U \times [0, \infty); t \cdot x \in U_l\}$ . The asymptotics (8.4) and Lemma 8.3 ensure us there is a smooth map  $\Theta_1 : \mathcal{U}_1 \rightarrow \mathbb{R}^n$  such that

$$(8.5) \quad t \cdot \phi(t \cdot x) = \hat{\phi}(x) + t^m \Theta_1(x, t) \quad \text{for all } (x, t) \in \mathcal{U}_1.$$

Set  $\psi = \phi^{-1}$ ; this is a smooth diffeomorphism from  $U_2$  onto  $U_1$ . In addition, let  $j \in \{1, \dots, n\}$ . By Lemma 5.4 there are smooth functions  $R_{j\alpha} \in C^\infty(U_2)$ ,  $|\alpha| \leq w_j + m \leq \langle \alpha \rangle$ , such that on  $U_2$  we have

$$(8.6) \quad \psi_j(x) = \sum_{\langle \alpha \rangle < w_j + m} \frac{1}{\alpha!} x^\alpha \partial_x^\alpha \psi_j(0) + \sum_{|\alpha| \leq w_j + m \leq \langle \alpha \rangle} x^\alpha R_{j\alpha}(x).$$

For  $k = -r, \dots, -w_j - 1$  set  $\psi_j^{[k]}(x) = 0$ , and, for  $k = -w_j, \dots, m - 1$  set

$$\psi^{[k]}_j(x) = \sum_{\langle \alpha \rangle = w_j + k} \frac{1}{\alpha!} x^\alpha \partial_x^\alpha \psi_j(0).$$



We observe that  $t^{-w_j}\psi^{[k]}_j(t \cdot x) = t^k\psi^{[k]}_j(x)$ . Thus, using (8.6) we obtain

$$t^{-w_j}\psi_j(t \cdot x) = x_j + \sum_{-r \leq k < m} t^k\psi^{[k]}_j(x) + O(t^m) \quad \text{as } t \rightarrow 0^+.$$

Therefore, if we set  $\psi^{[k]}(x) = (\psi^{[1]}(x), \dots, \psi^{[n]}(x))$ . Then we have

$$t^{-1} \cdot \psi(t \cdot x) = x + \sum_{-r \leq k < m} t^k\psi^{[k]}(x) + O(t^m) \quad \text{as } t \rightarrow 0^+.$$

It then follows from Lemma 8.1 that there is a smooth map  $\Theta_2 : \mathcal{U}_2 \rightarrow \mathbb{R}^n$  such that

$$(8.7) \quad t^{-1} \cdot \psi(t \cdot x) = x + \sum_{-r \leq k < m} t^k\psi^{[k]}(x) + t^m\Theta_2(x, t) \quad \text{for all } (x, t) \in \mathcal{U}_2.$$

Let  $x \in \mathbb{R}^n$  and  $t > 0$  be such that  $t \cdot x \in U_1$ . Note that

$$x = (\delta_t^{-1} \circ \psi \circ \delta_t) \circ (\delta_t^{-1} \circ \phi \circ \delta_t)(x) = t^{-1} \cdot \psi [t \cdot (t^{-1} \cdot \phi(t \cdot x))].$$

Combining this with (8.5) and (8.7) we obtain

$$\begin{aligned} x &= \sum_{-r \leq k < m} t^k\psi^{[k]}(t^{-1} \cdot \phi(t \cdot x)) + t^m\Theta_2(t^{-1} \cdot \phi(t \cdot x), t) \\ &= \sum_{-r \leq k < m} t^k\psi^{[k]}(\hat{\phi}(x) + t^m\Theta_1(x, t)) + t^m\Theta_2(\hat{\phi}(x) + t^m\Theta_1(x, t), t). \end{aligned}$$

As  $\psi^{[k]}(x + t^m\Theta_1(x, t)) = \psi^{[k]} \circ \hat{\phi}(x) + O(t^m)$  as  $t \rightarrow 0^+$ , we deduce that

$$x = \sum_{-r \leq k < m} t^k\psi^{[k]} \circ \hat{\phi}(x) + O(t^m) \quad \text{as } t \rightarrow 0^+.$$

It then follows that  $\psi^{[0]} \circ \hat{\phi}(x) = x$  and  $\psi^{[k]} \circ \hat{\phi}(x) = 0$  for  $k = -r, \dots, -m-1, k \neq 0$ . Combining this with (8.7) we then get

$$t^{-1} \cdot \psi(t \cdot x) = \hat{\phi}^{-1}(x) + t^m\Theta_2(x, t) \quad \text{for all } (x, t) \in \mathcal{U}_2.$$

By Lemma 8.3 this shows that  $\psi(x) = \hat{\phi}^{-1}(x) + O_w(\|x\|^{w+m})$  near  $x = 0$ . This proves the lemma.  $\square$

**Definition 8.5.** Let  $V$  be an open neighborhood of  $a$  in  $M$  and  $\varepsilon : V \rightarrow \mathbb{R}^n$  a smooth map such that  $\varepsilon(a) = 0$ . In addition, for  $j = 1, 2$  let  $\phi_j : V \rightarrow \mathbb{R}^{n'}$  be a smooth map. Given  $m \in \mathbb{N}$ , we shall write

$$(8.8) \quad \phi_1(x) = \phi_2(x) + O_w(\|\varepsilon(x)\|^{w+m}) \quad \text{near } x = a$$

when near  $x = a$  we may write

$$\phi_1(x) = \phi_2(x) + \Theta \circ \varepsilon(x),$$

where  $\Theta(x)$  is  $O_w(\|x\|^{w+m})$  near  $x = 0$ .

In case  $\varepsilon$  is a diffeomorphism near  $x = a$  we have the following characterization of (8.8).

**Lemma 8.6.** Let  $\varepsilon$  and  $\phi_j$ ,  $j = 1, 2$ , be as in Definition 8.5. Assume further that  $\varepsilon(x)$  is a diffeomorphism near  $x = a$ . Then the following are equivalent.

- (i)  $\phi_1(x) = \phi_2(x) + O_w(\|\varepsilon(x)\|^{w+m})$  near  $x = a$ .
- (ii)  $\phi_1 \circ \varepsilon^{-1}(x) = \phi_2 \circ \varepsilon^{-1}(x) + O_w(\|x\|^{w+m})$  near  $x = 0$ .

Combining Lemma 8.6 with Lemma 8.4 we then obtain the following result.

**Lemma 8.7.** Given  $m \in \mathbb{N}$ , let  $\phi_1$  and  $\phi_2$  be diffeomorphisms from an open neighborhood of  $a$  onto an open neighborhood of the origin  $0 \in \mathbb{R}^n$ . Then the following are equivalent:

- (i)  $\phi_2(x) = \phi_1(x) + O_w(\|\phi_1(x)\|^{w+m})$  near  $x = a$ .
- (ii)  $\phi_1(x) = \phi_2(x) + O_w(\|\phi_2(x)\|^{w+m})$  near  $x = a$ .

## 9. CHARACTERIZATION OF CARNOT COORDINATES

As it turns out, Carnot coordinates are by no means unique. In this section, we shall give a simple characterization of these coordinates and show how to obtain all such coordinates at a given point. Among various consequences, we will obtain a characterization of the Carnot map and compute this map in the case of Carnot-type groups.

In what follows, given a point  $a \in M$ , we let  $(X_1, \dots, X_n)$  an  $H$ -frame near  $a$ .

**9.1. Characterization of Carnot coordinates.** Keeping in mind the notation from Section 8, we have the following characterization of Carnot coordinates.

**Theorem 9.1.** *Let  $(x_1, \dots, x_n)$  be local coordinates near  $a$ .*

- (1) *A smooth coordinate change  $x \rightarrow \phi(x)$  near  $a$  provide us with Carnot coordinates at  $a$  adapted to the  $H$ -frame  $(X_1, \dots, X_n)$  if and only if we have*

$$(9.1) \quad \phi(x) = \varepsilon_a(x) + O_w(\|\varepsilon_a(x)\|^{w+1}).$$

- (2) *The assignment  $\psi \rightarrow \psi \circ \varepsilon_a^X$  gives a one-to-one correspondence between the following:*  
 (i) *Smooth maps  $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  that are  $O_w(\|x\|^{w+1})$ -perturbations of the identity map.*  
 (ii) *Coordinate changes  $x \rightarrow \phi(x)$  providing us with Carnot coordinates at  $a$  adapted to  $(X_1, \dots, X_n)$ .*

*Proof.* In what follows we denote by  $U$  the range of the local coordinates  $(x_1, \dots, x_n)$ . Suppose that the coordinates  $y = \phi(x)$  are Carnot coordinates at  $a$  adapted to the  $H$ -frame  $(X_1, \dots, X_n)$ . In particular, they are privileged coordinates at  $a$ , and so, for  $j = 1, \dots, n$ , the coordinate  $y_j = \phi_j(x)$  has order  $w_j$ . Therefore, by Proposition 4.6 in the privileged coordinates provided by  $\varepsilon_a$  the component  $\phi_j$  has weight  $w_j$ . This means that, near  $x = 0$ , we have

$$(9.2) \quad \phi_j \circ \varepsilon_a^{-1}(x) = \sum_{\langle \alpha \rangle = w_j} a_{j\alpha} x^\alpha + O(\|x\|^{w_j+1}), \quad a_{j\alpha} \in \mathbb{R}.$$

Set  $\hat{\phi}_j(x) = \sum_{\langle \alpha \rangle = w_j} a_{j\alpha} x^\alpha$ . Then  $\hat{\phi} := (\hat{\phi}_1, \dots, \hat{\phi}_n)$  is a  $w$ -homogeneous local diffeomorphism near the origin, and so it is a diffeomorphism on all  $\mathbb{R}^n$ . Moreover, the asymptotics (9.2) implies that, as  $t \rightarrow 0^+$ , we have

$$(9.3) \quad \delta_t^{-1} \circ [\phi \circ \varepsilon_a^{-1}] \circ \delta_t(x) = \hat{\phi}(x) + O(t) \quad \text{in } C^\infty(\varepsilon_a(U), \mathbb{R}^n).$$

Using Lemma 8.3 and Lemma 8.4 we see that, as  $t \rightarrow 0^+$ , we also have

$$(9.4) \quad \delta_t^{-1} \circ [\varepsilon_a \circ \phi^{-1}] \circ \delta_t(x) = \hat{\phi}^{-1}(x) + O(t) \quad \text{in } C^\infty(\phi(U), \mathbb{R}^n).$$

As  $\varepsilon_a$  and  $\phi$  provides us with Carnot coordinates at  $a$ , for  $j = 1, \dots, n$  and as  $t \rightarrow 0^+$ , we have

$$(9.5) \quad t^{w_j} \delta_t^* ((\varepsilon_a)_* X_j) = X_j^a + O(t) \quad \text{and} \quad t^{w_j} \delta_t^* (\phi_* X_j) = X_j^a + O(t),$$

where the asymptotics hold in  $\mathcal{X}(\varepsilon_a(U))$  and  $\mathcal{X}(\phi(U))$ , respectively. Combining the first asymptotics with (9.3)–(9.4) we deduce that, as  $t \rightarrow 0^+$  and in  $\mathcal{X}(\phi(U))$ , we also have

$$t^{w_j} \delta_t^* (\phi_* X_j) = (\delta_t^{-1} \circ (\phi \circ \varepsilon_a^{-1}) \circ \delta_t)_* [t^{w_j} \delta_t^* ((\varepsilon_a)_* X_j)] = \hat{\phi}_* X_j^a + O(t).$$

Therefore, we see that  $\hat{\phi}_* X_j^a = X_j^a$  for  $j = 1, \dots, n$ . It then follows from Lemma 6.5 that  $\hat{\phi} = \text{id}$ . Combining this with (9.2) shows that  $\phi \circ \varepsilon_a^{-1}(x) = x + O_w(\|x\|^{w+1})$  near  $x = 0$ . Using Lemma 8.6 we then deduce that  $\phi(x) = \varepsilon_a(x) + O_w(\|\varepsilon_a(x)\|^{w+1})$  near  $x = a$ .

Conversely, assume that

$$(9.6) \quad \phi(x) = \varepsilon_a(x) + O_w(\|\varepsilon_a(x)\|^{w+1}) \quad \text{near } x = a.$$

Then  $\phi'(a) = \varepsilon'_a(a) = \text{id}$ , and so  $\phi_* X_j(0) = (\varepsilon_a)^* X_j(0) = \partial_j$  for  $j = 1, \dots, n$ . Thus the coordinates  $y = \phi(x)$  are linearly adapted at  $a$  to the  $H$ -frame  $(X_1, \dots, X_n)$ .

Combining the asymptotics (9.6) with Lemma 8.6 and Lemma 8.3 shows that, as  $t \rightarrow 0^+$ , we have

$$t^{-1} \cdot (\phi \circ \varepsilon_a^{-1})(t \cdot x) = x + O(t) \quad \text{in } C^\infty(\varepsilon_a(U), \mathbb{R}^n).$$

Thanks to Lemma 8.4 the asymptotics (9.6) also implies that  $\varepsilon_a(x) = \phi(x) + O_w(\|\phi(x)\|^{w+1})$  near  $x = a$ . Therefore, as above we see that, as  $t \rightarrow 0^+$ , we have

$$t^{-1} \cdot (\varepsilon_a \circ \phi^{-1})(t \cdot x) = x + O(t) \quad \text{in } C^\infty(\phi(U), \mathbb{R}^n).$$

In addition, the first asymptotics in (9.5) still holds, since  $\varepsilon_a$  provides us with Carnot coordinates. Combining all this we deduce that, for  $j = 1, \dots, n$  and as  $t \rightarrow 0^+$ , we have

$$t^{w_j} \delta_t^*(\phi_* X_j) = (\delta_t^{-1} \circ (\phi \circ \varepsilon_a^{-1}) \circ \delta_t)_* [t^{w_j} \delta_t^*((\varepsilon_a)_* X_j)] = t^{w_j} \delta_t^*(\varepsilon_a)_* X_j + O(t) = X_j^a + O(t),$$

where the asymptotics hold in  $\mathcal{X}(\phi(U))$ . Using Proposition 7.2 we then deduce that  $\phi$  provides us with Carnot coordinates at  $a$ . This completes the first part of the theorem.

Finally, combining the first part with Lemma 8.6 we see that a coordinate change  $x \rightarrow \phi(x)$  provides us with Carnot coordinates at  $a$  adapted to  $(X_1, \dots, X_n)$  if and only if

$$\phi \circ (\varepsilon_a^X)^{-1}(x) = x + O_w(\|x\|^{w+1}) \quad \text{near } x = 0.$$

That is, the map  $x \rightarrow \phi(x)$  gives Carnot coordinates at  $a$  adapted to  $(X_1, \dots, X_n)$  if and only if the transition map  $\psi(x) = \phi \circ (\varepsilon_a^X)^{-1}(x)$  is a  $O_w(\|x\|^{w+1})$ -perturbations of the identity map. This proves the second part and completes the proof.  $\square$

As an application of Theorem 9.1 we have the following characterization of the normal Carnot coordinates.

**Theorem 9.2.** *The change of coordinates  $x \rightarrow \varepsilon_a(x)$  is the unique smooth change of coordinates such that*

- (i) *It provides us with Carnot coordinates at  $a$  adapted to the  $H$ -frame  $(X_1, \dots, X_n)$ .*
- (ii) *It is of the form  $x \rightarrow (\hat{\varepsilon} \circ T)(x)$ , where*
  - (a)  *$T$  is an affine map such that  $T(a) = 0$ .*
  - (b)  *$\hat{\varepsilon}$  is a polynomial diffeomorphism of the form,*

$$(9.7) \quad \hat{\varepsilon}_j(x) = x_j + \sum_{\substack{\langle \alpha \rangle \leq w_j \\ |\alpha| \geq 2}} d_{j\alpha} x^\alpha, \quad j = 1, \dots, n,$$

where the  $d_{j\alpha}$  are some constants.

*Proof.* Let  $x \rightarrow \phi(x)$  be a coordinate change satisfying (i)–(ii). Let us write  $\phi(x) = (\hat{\phi} \circ T)(x)$ , where  $T$  is an affine map such that  $T(a) = 0$  and  $\hat{\phi}$  is a polynomial diffeomorphism of the form (9.7). In the same way as in the proof of Proposition 4.19 it can be shown that  $T = T_a$ . Moreover, by Remark 7.12 we know that  $\varepsilon_a(x) = \hat{\varepsilon}_a \circ T_a(x)$ , where  $\hat{\varepsilon}_a(x) := \varepsilon_a \circ \psi_a(x)$  is a polynomial diffeomorphism of the form (9.7). Therefore, in order to complete the proof we only need to show that  $\hat{\phi} = \hat{\varepsilon}_a$ .

Using Theorem 9.1 we see that, near  $a$ , we have

$$\hat{\phi}(T_a(x)) = \phi(x) = \varepsilon_a(x) + O_w(\|\varepsilon_a(x)\|^{w+1}) = \hat{\varepsilon}_a(T_a(x)) + O_w(\|\hat{\varepsilon}_a(T_a(x))\|^{w+1}).$$

Therefore, near  $x = 0$ , we have

$$(9.8) \quad \hat{\phi} \circ \hat{\varepsilon}_a^{-1}(x) = x + O_w(\|x\|^{w+1}).$$

By Lemma 8.3 this means that  $(\hat{\phi} \circ \hat{\varepsilon}_a^{-1})_j(x) - x_j$  either is 0 or has weight  $\geq w_j + 1$  for  $j = 1, \dots, n$ , and so it could not have weight  $\leq w_j$  if it were to be non-zero.

Bearing this in mind, we observe that the polynomial diffeomorphisms of the form (9.7) form a subgroup of the diffeomorphism group of  $\mathbb{R}^n$ . In particular, the composite  $\hat{\phi} \circ \hat{\varepsilon}_a^{-1}$  is such a diffeomorphism. This implies that, for  $j = 1, \dots, n$ , the function  $(\hat{\phi} \circ \hat{\varepsilon}_a^{-1})_j(x) - x_j$  either is 0 or has weight  $\leq w_j$ . Since it could not have weight  $\leq w_j$  if it were non-zero, we deduce it must be zero. It then follows that  $\hat{\phi} \circ \hat{\varepsilon}_a^{-1}(x) = x$ , i.e.,  $\hat{\phi}(x) = \hat{\varepsilon}_a(x)$ . The proof is complete.  $\square$

**9.2. Normal Carnot coordinates on a Carnot-type group.** In order to better understand the map  $\varepsilon_a$  it is worth computing it in the special case of an  $n$ -dimensional Carnot-type group  $G$  equipped with a left-invariant Carnot structure. The Lie algebra  $\mathfrak{g}$  of  $G$  then has a grading  $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$  satisfying (2.2). For  $w = 1, \dots, r$  and  $j = \dim \mathfrak{h}_{w-1} + 1, \dots, \dim \mathfrak{h}_w$  set  $w_j = w$  and  $\mathfrak{h}_w = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_w$ . Let  $\{e_1, \dots, e_n\}$  be a basis of  $\mathfrak{g}$  such that  $e_j \in \mathfrak{g}_{w_j}$  for  $j = 1, \dots, n$ . We have structure constants  $L_{ij}^k$ ,  $w_i + w_j = w_k$ , so that

$$(9.9) \quad [e_i, e_j] = \begin{cases} \sum_{w_k=w_i+w_j} L_{ij}^k e_k & \text{if } w_i + w_j \leq r, \\ 0 & \text{if } w_i + w_j > r. \end{cases}$$

The basis  $\{e_j\}$  enables us to identify  $\mathfrak{g}$  with  $\mathbb{R}^n$  equipped with the Lie algebra structure defined by the relations (9.9) and the grading associated with the dilations (5.1). We shall assume that the exponential map is the identity, so as to identify  $G$  with  $\mathbb{R}^n$  equipped with a group law of the form,

$$(9.10) \quad (x \cdot y)_j = x_j + y_j + \sum_{\substack{\langle \alpha \rangle + \langle \beta \rangle = w_j \\ \alpha \neq 0, \beta \neq 0}} b_{j\alpha\beta} x^\alpha y^\beta,$$

where the constants  $b_{j\alpha\beta}$  satisfy (2.21). This ensures us that  $x^{-1} = -x$  for all  $x \in \mathbb{R}^n$ .

For  $j = 1, \dots, n$ , let  $X_j$  be the unique left-invariant vector field on  $G$  that agrees with  $\partial_j$  at  $x = 0$ . Equivalently, under the identification  $\mathfrak{g} \simeq TG(0)$ , this is the unique left-invariant vector field such that  $X_j = e_j$  at  $j = 0$ . We note that the vector fields  $X_j$  then satisfy the relations (9.9). In addition, for  $w = 1, \dots, r$  we denote by  $H_w$  the subbundle of  $TG$  generated by the vector fields  $X_j$ ,  $w_j \leq w$ . This defines a left-invariant Carnot flag  $H = (H_1, \dots, H_r)$  on  $G$  with respect to which  $(X_1, \dots, X_n)$  is an  $H$ -frame.

Let  $a \in G$  and, for  $j = 1, \dots, n$ , let  $\dot{X}_j(a)$  the class of  $X_j(a)$  in  $\mathfrak{g}_{w_j}G(a) = H_{w_j}(a)/H_{w_j-1}(a)$ . Then  $\{\dot{X}_1(a), \dots, \dot{X}_n(a)\}$  is a basis of  $\mathfrak{g}G(a)$ . Moreover, as the vector fields  $X_j$  satisfy the relations (9.9), so do the  $\dot{X}_j(a)$ . Therefore, under the coordinates defined by the basis  $\{\dot{X}_j(a)\}$ , the Lie algebra  $\mathfrak{g}G(a)$  agrees with  $\mathfrak{g}$ . As the exponential maps of  $G$  and  $GG(a)$  are identity maps this also identifies  $GG(a)$  with  $G$ . Incidentally, we see that in these coordinates we have

$$(9.11) \quad X_j^a = X_j \quad \text{for } j = 1, \dots, n.$$

**Proposition 9.3.** *Let  $a \in G$ . Then*

$$\varepsilon_a^X(x) = a^{-1} \cdot x = (-a) \cdot x \quad \text{for all } x \in G.$$

*Proof.* For  $x_0 \in G$  let us denote by  $\lambda_{x_0}$  the left-translation by  $x_0$ . The proof then amounts to showing that  $\varepsilon_a^X = \lambda_{-a}$ . Let us first show that  $\lambda_{-a}$  provides us with Carnot coordinates.

The very definition of the vector field  $X_j$  implies that  $(\lambda_{-a})_* X_j(0) = X_j(0) = \partial_j$ , so we see that the coordinates  $y = \lambda_{-a}(x)$  are linearly adapted at  $a$ . Moreover, the homogeneity and left-invariance of the vector field  $X_j$  imply that

$$t^{w_j} \delta_t^* ((\lambda_{-a})_* X_j) = t^{w_j} \delta_t^* X_j = X_j.$$

Combining this with (9.11) we deduce that, for  $j = 1, \dots, n$ , we have

$$X_j^{(a)} = \lim_{t \rightarrow 0^+} t^{w_j} \delta_t^* ((\lambda_{-a})_* X_j) = X_j = X_j^a \quad \text{in } \mathcal{X}(\mathbb{R}^n).$$

Using Proposition 7.2 then shows that  $\lambda_{-a}$  provides us with Carnot coordinates at  $a$ .

Let us now check that the map  $\lambda_{-a}(x)$  is of the form given by Theorem 9.1. For  $j = 1, \dots, n$ , let  $\epsilon_j$  be the  $j$ -th vector of the canonical basis of  $\mathbb{R}^n$  (i.e., the coordinate vector of  $\dot{X}_j(a)$ ). Then, for all smooth functions  $f$  on  $\mathbb{R}^n$ , we have

$$X_j f(a) = \frac{d}{dt} f \circ \lambda_a(t\epsilon_j)|_{t=0} = \sum_k \partial_j \lambda_a(0)_k \partial_k f(a).$$

Therefore, the coefficient matrix of the frame  $(X_1, \dots, X_n)$  at  $a$  is equal to  $(\partial_j \lambda_a(0)_k) = \lambda'_a(0)^T$ . As the equality  $\lambda_{-a} \circ \lambda_a = \text{id}$  implies that  $\lambda'_{-a}(a) = \lambda'_a(0)^{-1}$ , we then deduce that the affine map  $T_a(x)$  is given by

$$T_a(x) = (\lambda'_a(0)^T)^{-T} (x - a) = \lambda'_{-a}(0)(x - a).$$

Note that these equalities are tantamount to the fact that  $\lambda_{-a}$  provides us with linearly adapted coordinates at  $a$ .

It follows from (9.10) that  $\lambda_a(x)_j$  is affine function which has slope 1 with respect to  $x_j$  and is independent of the variables  $x_k$  with  $w_k \geq w_j$  and  $k \neq j$ . Therefore, we see that  $\partial_k \lambda_a(0)_j = \delta_{jk}$  for  $w_k \geq w_j$ , and hence we have

$$(9.12) \quad (\lambda'_a(0)x)_j = x_j + \sum_{w_k < w_j} \partial_k \lambda_a(0)_j x_k \quad \text{for all } x \in \mathbb{R}^n.$$

Bearing this in mind, Eq. (9.10) shows that  $\lambda_{-a}(x)_j$  is a linear combination of monomials  $x^\alpha$  with  $\langle \alpha \rangle \leq w_j$ . Therefore, the Taylor formula for polynomials gives

$$(9.13) \quad \lambda_{-a}(x)_j = (\lambda'_{-a}(a)(x - a))_j + R_j(x - a), \quad R_j(y) := \sum_{\substack{\langle \alpha \rangle \leq w_j \\ |\alpha| \geq 2}} \frac{1}{\alpha!} \partial^\alpha \lambda_{-a}(a) y^\alpha.$$

Set  $\tilde{R}_j(x) = R_j \circ (\lambda'_a(0))(x)$ . Using (9.13) and the equality  $x - a = \lambda'_a(0)^{-1} \circ T_a(x)$ , we get

$$\lambda_{-a}(x) = T_a(x)_j + \tilde{R}_j \circ T_a(x).$$

Furthermore, using (9.12) and the expression of  $R_j(x)$  above, we see that  $\tilde{R}_j$  is a linear combination of monomials  $x^\alpha$  with  $\langle \alpha \rangle \leq w_j$  and  $|\alpha| \geq 2$ . Therefore, we see that  $\lambda_{-a}(x)$  is a polynomial map of the form (9.7). As it provides us with Carnot privileged coordinates we then deduce from Theorem 9.2 that  $\lambda_{-a} = \varepsilon_a^X$ . The proof is complete.  $\square$

## 10. DEPENDENCE ON THE $H$ -FRAME

In this section, we give a closer look at the dependence of the  $\varepsilon_a$ -map with respect to the base point  $a$ . We observe that, once we are given local coordinates  $x = (x_1, \dots, x_n)$ , the construction of the  $\varepsilon_a$ -map only involves the coefficients in these local coordinates of the vector fields of the  $H$ -frame  $X = (X_1, \dots, X_n)$ . As different choices of  $H$ -frames may produce different normal Carnot coordinate maps, we shall denote by  $\varepsilon_a^X$  the normal Carnot coordinate map at  $a$  associated with the  $H$ -frame  $X = (X_1, \dots, X_n)$ .

In order to study the dependence of  $\varepsilon_a^X$  with respect the  $H$ -frame  $X = (X_1, \dots, X_n)$  we shall denote by  $B_X(x)$  the matrix of the coefficients of this  $H$ -frame in the local coordinates  $x = (x_1, \dots, x_n)$ . That is,

$$B_X(x) = (b_{jk}(x)), \quad \text{where } X_j = \sum_{1 \leq k \leq n} b_{jk}(x) \partial_{x_k}.$$

As it follows from the proof of Lemma 4.8 the affine map  $T_a(x)$  is given by

$$T_a(x) = B_X(a)^{-T} (x - a),$$

where  $B_X(a)^{-T}$  is the inverse transpose matrix of  $B_X(a)$ .

**Proposition 10.1.** *For  $j = 1, \dots, n$ , we have*

$$(10.1) \quad \varepsilon_a^X(x)_j = T_a(x)_j + \sum_{\substack{\langle \alpha \rangle \leq w_j \\ |\alpha| \geq 2}} c_{j\alpha}(B_X(a)) T_a(x)^\alpha,$$

where  $T_a(x) = B_X(a)^{-T} (x - a)$  and  $c_{j\alpha}(B_X(a))$  is a universal polynomial in the coefficients of the matrices  $B_X(a)^{-T}$  and  $\partial_x^\beta B_X(a)$  with  $|\beta| \leq |\alpha| - 1$ .

*Proof.* The proof is a consequence of a series of reductions. First, in order to distinguish between the various types of coordinates at stake we set

$$\bar{x} = T_a(x), \quad \tilde{x} = \psi_a(\bar{x}) = \psi_a \circ T_a(x), \quad \hat{x} = \hat{\varepsilon}_a(\tilde{x}) = \varepsilon_a(x).$$

Thus,  $\bar{x}$  (resp.,  $\tilde{x}$ ,  $\hat{x}$ ) are the linearly adapted (resp., privileged, Carnot privileged) coordinates defined by the map  $T_a$  (resp.,  $\psi_a \circ T_a$ ,  $\varepsilon_a$ ). We also set

$$(T_a)_* X_j = \sum_{1 \leq j \leq n} \bar{b}_{jk}(\bar{x}) \partial_{\bar{x}_k} \quad \text{and} \quad (\psi_a \circ T_a)_* X_j = \sum_{1 \leq j \leq n} \tilde{b}_{jk}(\tilde{x}) \partial_{\tilde{x}_k}.$$

In other words, the coefficients  $\bar{b}_{jk}(\bar{x})$  (resp.,  $\tilde{b}_{jk}(\tilde{x})$ ) are the coefficients of the vector field  $X_j$  in the coordinates  $\bar{x}$  (resp.,  $\tilde{x}$ ). We note that as  $T_a x = B_X(a)^{-T}(x - a)$ , the coefficients  $b_{jk}(x)$  and  $\bar{b}_{jk}(\bar{x})$  are related by

$$\bar{b}_{jk}(\bar{x}) = \sum_{1 \leq j \leq l} b_{jl} (B_X(a)^T(x - a)) B_X(a)_{kl}^{-T}.$$

It then follows that each partial derivative  $\partial_{\bar{x}}^\alpha \bar{b}_{jk}(0)$  is of the form,

$$(10.2) \quad \partial_{\bar{x}}^\alpha \bar{b}_{jk}(0) = \sum_{1 \leq l \leq n} \sum_{|\beta|=\alpha} d_{jl\alpha} \partial_x^\beta b_{jl}(a) B_X(a)_{kl}^{-T},$$

where  $d_{jl\alpha}$  is a universal polynomial in the coefficients of the matrix  $B_X(a)^T$ .

Bearing this in mind, thanks to Remark 7.12 we know that

$$\varepsilon_a(x)_j = T_a(x)_j + \sum_{\substack{\langle \alpha \rangle \leq w_j \\ |\alpha| \geq 2}} c_{j\alpha} T_a(x)^\alpha, \quad j = 1, \dots, n,$$

where the constants  $c_{j,\alpha}$  are as follows:

- For  $\langle \alpha \rangle = w_j$  they agree with the constants  $c_{j\alpha}$  in (7.2).
- For  $\langle \alpha \rangle < w_j$  they are universal polynomials in the constants  $c_{k\beta}$  and  $a_{l\gamma}$ , with  $\langle \beta \rangle = w_k < w_j$  and  $\langle \gamma \rangle < w_l < w_j$ , where the constants  $a_{l\gamma}$  are in (4.10).

Combining this with (10.2) we see that in order to complete the proof it is enough to show that each coefficient  $c_{j\alpha}$  is a universal polynomial in the partial derivatives  $\partial_{\bar{x}}^\beta \bar{b}_{kl}(0)$  with  $|\beta| \leq |\alpha| - 1$ . Furthermore, by Remark 4.17 and Lemma 7.5 we know that

- Each coefficient  $a_{j\alpha}$  is a universal polynomial in the partial derivatives  $\partial_{\bar{x}}^\beta \bar{b}_{kl}(0)$  with  $|\beta| \leq |\alpha| - 1$ .
- Each coefficient  $c_{j\alpha}$ ,  $\langle \alpha \rangle = w_j$ , is a universal polynomial in the partial derivatives  $\partial_{\bar{x}}^\beta \tilde{b}_{kl}(0)$  with  $|\beta| \leq \langle \beta \rangle < w_j - w_l \leq |\alpha| - 1$ .

Therefore, in order to complete the proof we only have to prove that each partial derivative  $\partial_{\bar{x}}^\alpha \tilde{b}_{jk}(0)$  is a universal polynomial in the partial derivatives  $\partial_{\bar{x}}^\beta \bar{b}_{lp}(0)$  with  $|\beta| \leq |\alpha|$ . To see this we note that the coefficients  $\tilde{b}_{jk}(\tilde{x})$  and  $\bar{b}_{jk}(\bar{x})$  are related by

$$\tilde{b}_{jk}(\tilde{x}) = \sum_{1 \leq l \leq n} \bar{b}_{jl}(\psi_a^{-1}(x)) (\partial_{\bar{x}_l} \psi_a)(\psi_a^{-1}(x))_k.$$

We then deduce that each partial derivative  $\partial_{\bar{x}}^\alpha \tilde{b}_{jk}(0)$  is a universal polynomial in the following

- The partial derivatives  $\partial_{\bar{x}}^\beta \bar{b}_{jl}(0)$  with  $|\beta| \leq |\alpha|$ .
- The partial derivatives  $\partial_{\bar{x}}^\gamma \psi_a(0)_k$  with  $|\gamma| \leq |\alpha| + 1$ .
- The partial derivatives  $\partial_{\bar{x}}^\gamma \psi_a^{-1}(0)_k$  with  $|\gamma| \leq |\alpha|$ .

This further reduces the proof to showing that the partial derivative  $\partial_{\bar{x}}^\alpha \psi_a(0)_j$  and  $\partial_{\bar{x}}^\alpha \psi_a^{-1}(0)_j$  are universal polynomials in the partial derivatives  $\partial_{\bar{x}}^\beta \bar{b}_{kl}(0)$  with  $|\beta| \leq |\alpha| - 1$ .

Having said this, the partial derivatives  $\partial_{\bar{x}}^\alpha \psi_a(0)_j$  can be computed explicitly. Indeed, we have

$$(10.3) \quad \psi_a(\bar{x})_j = \bar{x}_j + \sum_{\substack{\langle \alpha \rangle < w_j \\ |\alpha| \geq 2}} a_{j\alpha} \bar{x}^\alpha, \quad j = 1, \dots, n.$$

Thus  $\psi'_a(0) = \text{id}$  and, for  $|\alpha| \geq 2$ , we have

$$\partial_x^\alpha \psi_a(0)_j = \begin{cases} \alpha! a_{j\alpha} & \text{if } \langle \alpha \rangle < w_j \\ 0 & \text{otherwise.} \end{cases}$$

As  $a_{j\alpha}$  is a universal polynomial in the partial derivatives  $\partial_x^\beta \bar{b}_{kl}(0)$  with  $|\beta| \leq |\alpha| - 1$ , we then deduce that so is  $\partial_x^\alpha \psi_a(0)_j$ . In addition, by using (10.3) and arguing by induction it can be shown that  $\psi_a^{-1}(\tilde{x})$  is of the form,

$$\psi_a^{-1}(\tilde{x})_j = \tilde{x}_j + \sum_{\substack{\langle \alpha \rangle < w_j \\ |\alpha| \geq 2}} \tilde{a}_{j\alpha}(\tilde{x})^\alpha,$$

where  $\tilde{a}_{j\alpha}$  is a universal polynomial in the coefficients  $a_{k\beta}$  with  $w_k \leq w_j$  and  $|\beta| \leq |\alpha|$ , and hence is a universal polynomial in the partial derivatives  $\partial_x^\beta \bar{b}_{kl}(0)$  with  $|\beta| \leq |\alpha| - 1$ . Therefore, by arguing as above it can be shown that each partial derivative  $\partial_x^\alpha \psi_a^{-1}(0)_j$  is a universal polynomial in the partial derivatives  $\partial_x^\beta \bar{b}_{kl}(0)$  with  $|\beta| \leq |\alpha| - 1$ . This completes the proof.  $\square$

As it turns out, for our purpose we will also need a version of Proposition 10.1 for the inverse map  $(\varepsilon_a^X)^{-1}(x)$ .

**Proposition 10.2.** *The inverse map  $(\varepsilon_a^X)^{-1}(x)$  is of the form,*

$$(10.4) \quad \begin{aligned} (\varepsilon_a^X)^{-1}(x) &= B_X(a)^T \tilde{\varepsilon}_a(x) + a, \\ \tilde{\varepsilon}_a(x)_j &= x_j + \sum_{\substack{\langle \alpha \rangle \leq w_j \\ |\alpha| \geq 2}} d_{j\alpha}(B_X(a)) x^\alpha, \quad j = 1, \dots, n, \end{aligned}$$

where  $d_{j\alpha}(B_X(a))$  is a universal polynomial in the coefficients of the matrices  $B_X(a)^{-T}$  and  $\partial_x^\beta B_X(a)$  with  $|\beta| \leq |\alpha| - 1$ .

*Proof.* Thanks to Proposition 10.1 we know that  $\varepsilon_a^X = \hat{\varepsilon}_a \circ T_a(x)$ , where  $T_a(x) = B_X(a)^{-T}(x - a)$  and

$$(10.5) \quad \hat{\varepsilon}_a(x)_j = x_j + \sum_{\substack{\langle \alpha \rangle \leq w_j \\ |\alpha| \geq 2}} c_{j\alpha}(B_X(a)) x^\alpha, \quad j = 1, \dots, n,$$

where  $c_{j\alpha}(B_X(a))$  is a universal polynomial in the coefficients of the matrices  $B_X(a)^{-T}$  and  $\partial_x^\beta B_X(a)$  with  $|\beta| \leq |\alpha| - 1$ . Set  $\tilde{\varepsilon}_a = (\hat{\varepsilon}_a)^{-1}$ . Then we have

$$(\varepsilon_a^X)^{-1}(x) = T_a^{-1} \circ \tilde{\varepsilon}_a(x) = B_X(a)^T \tilde{\varepsilon}_a(x) + a.$$

Note that (10.5) shows that  $\hat{\varepsilon}_a(x)_j - x_j$  vanishes identically for  $w_j = 1$  and is a polynomial in the variables  $x_k$  with  $w_k < w_j$  for  $w_j \geq 2$ . Therefore, a simple induction shows that, for  $j = 1, \dots, n$ , we have

$$\tilde{\varepsilon}_a(x)_j = x_j + \sum_{\substack{\langle \alpha \rangle \leq w_j \\ |\alpha| \geq 2}} d_{j\alpha} x^\alpha,$$

where  $d_{j\alpha}$  is a universal polynomial in the  $c_{k\beta}(B_X(a))$  with  $\langle \beta \rangle \leq w_k \leq w_j$ , and hence is a universal polynomial in the coefficients of the matrices  $B_X(a)^{-T}$  and  $\partial_x^\beta B_X(a)$  with  $|\beta| \leq |\alpha| - 1$ . The result is thus proved.  $\square$

As immediate consequences of Proposition 10.1 and Proposition 10.2 we obtain the following smoothness result.

**Proposition 10.3.** *Given an  $H$ -frame  $X = (X_1, \dots, X_n)$  over a coordinate open  $U \subset \mathbb{R}^n$ , the maps  $(y, x) \rightarrow \varepsilon_x(y) \in \mathbb{R}^n$  and  $(y, x) \rightarrow \varepsilon_x^{-1}(y)$  are smooth maps from  $U \times \mathbb{R}^n$  to  $\mathbb{R}^n$ .*

Let  $U$  be an open subset of  $\mathbb{R}^n$  and denote by  $\mathcal{H}(U)$  the subspace of  $\mathcal{X}(U)^n$  consisting of frames  $(X_1, \dots, X_n)$  such that, for  $i, j = 1, \dots, n$  with  $w_i + w_j \leq r$ , we have

$$[X_i, X_j](x) \in \text{Span} \{X_k(x); w_k \leq w_i + w_j\} \quad \text{for all } x \in U.$$



We note that if  $X = (X_1, \dots, X_n)$  is in  $\mathcal{H}(U)$ , then this is an  $H$ -frame with respect to the flag  $H = (H_1, \dots, H_r)$ , where  $H_w$  is the sub-bundle generated by the vector fields  $X_j$ ,  $w_j \leq w$ .

We shall equip  $\mathcal{H}(U)$  with the topology induced by that of  $\mathcal{X}(U)^n$ . We observe that in (10.1) and (10.4) the coefficients  $c_{j\alpha}(B_X(a))$  and  $d_{j\alpha}(B_X(a))$  depend continuously on  $X = (X_1, \dots, X_n)$  with respect to the aforementioned topology on  $\mathcal{H}(U)$ . Therefore, we arrive at the following statement.

**Proposition 10.4.** *The maps  $X \rightarrow \varepsilon_y^X(x)$  and  $X \rightarrow (\varepsilon_y^X)^{-1}(x)$  are continuous from  $\mathcal{H}(U)$  to  $C^\infty(U \times \mathbb{R}^n, \mathbb{R}^n)$ .*

## 11. TANGENT APPROXIMATION OF CARNOT MANIFOLD MAPS

In this section, we shall show that, in Carnot coordinates, a Carnot manifold map is approximated in a very precise way by its tangent map defined in Section 2. Let  $(M', H')$  be a step  $r$  Carnot manifold and  $\phi : M \rightarrow M'$  be a Carnot manifold map. Thus  $H' = (H'_1, \dots, H'_r)$  and, for  $w = 1, \dots, r$ , we have

$$(11.1) \quad \phi'(x)X \in H'_w(\phi(x)) \quad \text{for all } (x, X) \in H_w.$$

Set  $n' = \dim M'$  and let  $w' = (w_1, \dots, w_{n'})$  be the weight sequence of  $(M', H')$  in the sense of Definition 2.10.

Given  $a \in M$ , set  $a' = \phi(a)$ , and let  $(X_1, \dots, X_n)$  be an  $H$ -frame near  $a$  and  $(X'_1, \dots, X'_{n'})$  an  $H'$ -frame near  $a'$ . The Carnot manifold map property (11.1) implies that, near  $x = a$ , we may write

$$(11.2) \quad \phi'(x)(X_j(x)) = \sum_{w'_k \leq w_j} c_{jk}(x) X'_k(\phi(x)), \quad j = 1, \dots, n,$$

for some smooth coefficients  $c_{jk}(x)$ .

Recall that the Carnot manifold map  $\phi$  gives rise to a Lie group bundle map  $\phi'_H : GM \rightarrow GM'$  (cf. Section 2). Moreover, the  $H$ -frame  $(X_1, \dots, X_n)$  yields a linear basis  $\{\dot{X}_j(a)\}_{1 \leq j \leq n}$  of  $\mathfrak{g}M(a)$ , and hence provides us with a global coordinate system on  $\mathfrak{g}M(a)$  and  $GM(a)$ . Likewise, the  $H'$ -frame  $(X'_1, \dots, X'_{n'})$  yields a basis  $\{\dot{X}'_k(a')\}_{1 \leq k \leq n'}$  of  $\mathfrak{g}M'(a')$  and provides us with a global coordinate system on  $\mathfrak{g}M'(a')$  and  $GM'(a')$ . This allows us to regard the vector fields  $X_j^a$ ,  $j = 1, \dots, n$ , and  $(X'_k)^{a'}$ ,  $k = 1, \dots, n'$  as vector fields on  $\mathbb{R}^n$  and  $\mathbb{R}^{n'}$ , respectively. In addition, we shall identify  $\phi'_H(a)$  with the linear map that it defines in these coordinates systems. This allows us to regard  $\phi'_H(a)$  as a linear map  $\phi'_H(a)x = (\phi'_H(a)_1x, \dots, \phi'_H(a)_{n'}x)$  from  $\mathbb{R}^n$  to  $\mathbb{R}^{n'}$ . Incidentally, this allows us to identify  $\phi'_H(a)$  with its differential.

**Lemma 11.1.** *Under the above identifications, the linear map  $\phi'_H(a)$  has the following properties.*

(1) *For  $k = 1, \dots, n'$ , we have*

$$(11.3) \quad \phi'_H(a)_k x = \sum_{w_j = w'_k} c_{jk}(a) x_j.$$

(2) *For  $j = 1, \dots, n$ , we have*

$$(11.4) \quad \phi'_H(a)(X_j^a(x)) = \sum_{w'_l = w_k} c_{jk}(a)(X'_l)^{a'}(\phi'_H(a)x) \quad \text{for all } x \in \mathbb{R}^n.$$

(3) *Assume that  $\phi$  is a Carnot diffeomorphism and  $X_j^l = \phi_* X_j$  for  $j = 1, \dots, n$ . Then  $\phi'_H(a)$  agrees with the identity map.*

*Proof.* Let  $(\phi'_H(a)_{kj})$  be the  $n' \times n$  matrix of  $\phi'_H(a)$ . As for  $w = 1, \dots, r$ , the map  $\phi'_H(a)$  maps  $\mathfrak{g}_w M(a)$  to  $\mathfrak{g}_w M'(a')$ , we see that, for  $j = 1, \dots, n$ , we have

$$\phi'_H(a)(\dot{X}_j(a)) = \sum_{1 \leq k \leq n'} \phi'_H(a)_{kj} \dot{X}'_k(a') = \sum_{w'_k = w_j} \phi'_H(a)_{kj} \dot{X}'_k(a').$$

Moreover, by the very definition of  $\phi'_H(a)$  we know that  $\phi'_H(a)(\dot{X}_j(a))$  is the class of  $[\phi'(a)(X_j(a))]$  modulo  $H'_{w_j-1}(a')$ . Therefore, using (11.2) we obtain

$$\phi'_H(a)(\dot{X}_j(a)) = \sum_{w'_k=w_j} c_{jk}(a) \dot{X}'_k(a').$$

It then follows that  $\phi'_H(a)_{kj} = c_{jk}(a)$  for  $w_j = w'_k$  and  $\phi'_H(a)_{kj} = 0$  for  $w_j \neq w'_k$ . Thus, for  $k = 1, \dots, n'$ , we have

$$(11.5) \quad \phi'_H(a)_k x = \sum_{1 \leq j \leq n} \phi'_H(a)_{kj} x_j = \sum_{w_j=w'_k} c_{jk}(a) x_k.$$

This proves the first part of the lemma. Identifying  $\phi'_H(a)$  with its differential, this also shows that, for  $j = 1, \dots, n$ , we have

$$(11.6) \quad \phi'_H(a)[X_j^{a'}(0)] = \phi'_H(a)[\partial_{x_j}] = \sum_{w'_k=w_j} c_{jk}(a) \partial_{x'_k} = \sum_{w'_k=w_j} c_{jk}(a) (X'_k)^{a'}(0).$$

Given  $y \in \mathbb{R}^n$ , we shall denote by  $\lambda_y^a$  the left-translation by  $y$  in  $GM(a)$ . Likewise, for  $y' \in \mathbb{R}^{n'}$ , we shall denote by  $\lambda_{y'}^{a'}$  the left-translation by  $y'$  in  $GM'(a')$ . The fact that  $\phi'_H(a)$  is a group morphism from  $GM(a)$  to  $GM'(a')$  implies that

$$\phi'_H(a) \circ \lambda_y^a = \lambda_{\phi'_H(a)y}^{a'} \circ \phi'_H(a) \quad \text{and} \quad \phi'_H(a) \circ [(\lambda_y^a)'(x)] = [(\lambda_{\phi'_H(a)y}^{a'})'(\phi'_H(a)x)] \circ \phi'_H(a).$$

Furthermore, the left-invariance of the vector fields  $X_j^a$ ,  $j = 1, \dots, n$ , and  $(X'_k)^{a'}$ ,  $k = 1, \dots, n'$ , implies that

$$X_j^a(y \cdot x) = (\lambda_y^a)'(x)[X_j(x)] \quad \text{and} \quad (X'_k)^{a'}(y' \cdot x') = (\lambda_{y'}^{a'})'(x')[X'_k(x')].$$

We then deduce that

$$\begin{aligned} \phi'_H(a)[X_j^a(x)] &= \phi'_H(a) \circ (\lambda_x^a)'(0)[X_j^a(0)] = (\lambda_{\phi'_H(a)x}^{a'})'(0) \circ \phi'_H(a)[X_j^a(0)], \\ (X'_k)^{a'}(\phi'_H(a)x) &= (X'_k)^{a'}(\lambda_{\phi'_H(a)x}^a(0)) = (\lambda_{\phi'_H(a)x}^{a'})'(0)[(X'_k)^{a'}(0)]. \end{aligned}$$

Combining this with the expression of  $\phi'_H(a)[X_j^a(0)]$  given by (11.6) we get

$$\phi'_H(a)[X_j^a(x)] = \sum_{w'_k=w_j} c_{jk}(a) (\lambda_{\phi'_H(a)x}^{a'})'(0)[(X'_k)^{a'}(0)] = \sum_{w'_k=w_j} c_{jk}(a) (X'_k)^{a'}(\phi'_H(a)x).$$

This proves the 2nd part of the lemma.

Finally, assume further that  $\phi$  is a Carnot diffeomorphism and  $X'_j = \phi_* X_j$  for  $j = 1, \dots, n$ . Then in (11.2) we have  $c_{jk}(x) = \delta_{jk}$ , and so by using (11.5) we see that in this case  $\phi'_H(a)$  agrees with the identity map. This proves the last part of the lemma and completes the proof.  $\square$

*Remark 11.2.* In case  $\phi$  is a diffeomorphism, the map  $\phi'_H(a)$  is a diffeomorphism of  $\mathbb{R}^n$ , and then (11.4) means that

$$(\phi'_H(a))_* X_j^a = \sum_{w'_k=w_j} c_{jk}(a) (X'_k)^{a'} \quad \text{for } j = 1, \dots, n.$$

In what follows, we let  $(x_1, \dots, x_n)$  be Carnot coordinates at  $a$  adapted to  $(X_1, \dots, X_n)$  and  $(x_1, \dots, x_{n'})$  Carnot coordinates at  $a'$  adapted to  $(X_1, \dots, X_{n'})$ . Using these coordinates we shall regard the map  $\phi$  as a smooth map  $\phi(x) = (\phi_1(x), \dots, \phi_{n'}(x))$  from an open neighborhood of the origin in  $\mathbb{R}^n$  to a neighborhood of the origin in  $\mathbb{R}^{n'}$ . Likewise, we will regard the vector fields  $X_j$ ,  $j = 1, \dots, n$ , and  $X'_k$ ,  $k = 1, \dots, n'$ , as vector fields on a neighborhood of the origin in  $\mathbb{R}^n$  and  $\mathbb{R}^{n'}$ , respectively. For  $j = 1, \dots, n$  and  $k = 1, \dots, n'$ , set

$$X_j = \sum_{1 \leq l \leq n} b_{jl}^X(x) \partial_{x_l} \quad \text{and} \quad X'_k = \sum_{1 \leq l \leq n'} b_{kl}^{X'}(x') \partial_{x'_l},$$

where the coefficients  $b_{jl}^X(x)$  (resp.,  $b_{kl}^{X'}(x')$ ) are smooth functions near the origin in  $\mathbb{R}^n$  (resp.,  $\mathbb{R}^{n'}$ ). Regarding the coefficients  $c_{jk}(x)$  as smooth functions near the origin  $0 \in \mathbb{R}^n$ , the equality (11.2) exactly means that, for  $j = 1, \dots, n$  and  $k = 1, \dots, n'$ , we have

$$(11.7) \quad \sum_{1 \leq l \leq n} b_{jl}^X(x) \partial_{x_l} \phi_k(x) = \sum_{\substack{1 \leq l \leq n' \\ w'_l \leq w_j}} c_{jl}(x) b_{lk}^{X'}(\phi(x)).$$

**Lemma 11.3.** *For  $k = 1, \dots, n$ , the component  $\phi_k(x)$  has weight  $\geq w'_k$ .*

*Proof.* We need to prove that

$$(11.8) \quad \partial_x^\alpha \phi_k(0) = 0 \quad \text{whenever } w'_k - \langle \alpha \rangle > 0.$$

We shall prove (11.8) by induction on  $|\alpha|$ . Let us say that (11.8) holds up to order  $m$  when it holds for every multi-order  $\alpha$  such that  $w'_k - \langle \alpha \rangle > 0$  and  $|\alpha| \leq m$ . We note that it holds up to order 0 since  $\phi(0) = 0$ .

It remains to show that if (11.8) holds up to order  $m$ , then it holds up to order  $m+1$ . To reach this end we observe that, as the coordinates  $(x_1, \dots, x_n)$  are Carnot coordinates at  $a$  adapted to the  $H$ -frame  $(X_1, \dots, X_n)$ , we know that the vector field  $X_j(0)$  has weight  $-w_j$  and agrees with  $\partial_{x_j}$  at  $x = 0$ . The former property means that each coefficient  $b_{jl}^X(x)$  has weight  $\geq w_l - w_j$ . Therefore, we see that, for  $j, l = 1, \dots, n$ , we have

$$(11.9) \quad b_{jl}^X(0) = \delta_{jl} \quad \text{and} \quad \partial_x^\alpha b_{jl}^X(0) = 0 \quad \text{whenever } w_l - w_j - \langle \alpha \rangle > 0.$$

Likewise, for  $l, k = 1, \dots, n'$ , we have

$$(11.10) \quad b_{lk}^{X'}(0) = \delta_{lk} \quad \text{and} \quad \partial_{x'}^\alpha b_{lk}^{X'}(0) = 0 \quad \text{whenever } w'_k - w'_l - \langle \alpha \rangle > 0.$$

Suppose now that (11.8) holds up to order  $m$ . Given  $j \in \{1, \dots, n\}$  and  $k \in \{1, \dots, n'\}$ , let  $\alpha \in \mathbb{N}_0^n$  be such that  $w'_k - w_j - \langle \alpha \rangle > 0$  and  $|\alpha| \leq m$ . Using the equality  $b_{jl}^X(0) = \delta_{jl}$ , we see that the partial derivative of order  $\alpha$  at  $x = 0$  of the l.h.s. of (11.7) is equal to

$$\partial_x^\alpha \partial_{x_j} \phi_k(0) + \sum_{\substack{\beta + \gamma = \alpha \\ \gamma \neq \alpha}} \binom{\alpha}{\beta} \partial_x^\beta b_{jl}^X(0) \partial_x^\gamma \partial_{x_l} \phi_k(0).$$

We claim that in the summation above each term  $\partial_x^\beta b_{jl}^X(0) \partial_x^\gamma \partial_{x_l} \phi_k(0)$  is zero. To see this we observe that

$$(w_l - w_j - \langle \beta \rangle) + (w_k - w_l - \langle \gamma \rangle) = w_k - w_j - \langle \alpha \rangle > 0.$$

Therefore, at least one the integers  $w_l - w_j - \langle \beta \rangle$  or  $w_k - w_l - \langle \gamma \rangle$  must be positive. If the former is positive, then (11.9) ensures us that  $\partial_x^\beta b_{jl}^X(0) = 0$ . If  $w_k - w_l - \langle \gamma \rangle > 0$ , then, as  $\gamma \neq \alpha$ , we have  $|\gamma| + 1 \leq (|\alpha| - 1) + 1 \leq m$ . As by assumption (11.8) holds up to order  $m$  we see that  $\partial_x^\gamma \partial_{x_l} \phi_k(0) = 0$ . In any case  $\partial_x^\beta b_{jl}^X(0) \partial_x^\gamma \partial_{x_l} \phi_k(0)$  must be zero. We then deduce that the partial derivative of order  $\alpha$  at  $x = 0$  of the l.h.s. of (11.7) is equal to  $\partial_x^\alpha \partial_{x_j} \phi_k(0)$ .

Bearing this in mind, the partial derivative of order  $\alpha$  at  $x = 0$  of the r.h.s. of (11.7) is equal to

$$(11.11) \quad \sum_{\substack{\beta + \gamma = \alpha \\ w'_l \leq w_j}} \binom{\alpha}{\beta} \partial_x^\beta c_{jl}(0) \partial_x^\gamma (b_{lk}^{X'} \circ \phi)(0).$$

*Claim.* Let  $\gamma \in \mathbb{N}_0^n$  and  $b \in C^\infty(U')$ . Then  $\partial_x^\gamma (b \circ \phi)(x)$  is a linear combination of terms of the form

$$(11.12) \quad \partial_x^{\gamma_1} \phi_{q_1}(x) \cdots \partial_x^{\gamma_p} \phi_{q_p}(x) (\partial_{x'_{q_1}} \cdots \partial_{x'_{q_p}} b)(\phi(x)),$$

where  $p$  ranges over  $\{1, \dots, |\gamma|\}$ , the integers  $q_1, \dots, q_p$  range over  $\{1, \dots, n'\}$  and  $(\gamma_1, \dots, \gamma_p)$  ranges over all  $p$ -uples in  $(\mathbb{N}_0^n)^p$  such that  $\gamma_1 + \cdots + \gamma_p = \gamma$ .

*Proof.* This claim is a by-product of the multivariate higher-order chain rule (a.k.a. multivariate Faà de Bruno formula). For our purpose we don't need the precise expressions of the coefficients of the various terms (11.12). We thus can proceed to prove the claim by induction on  $|\gamma|$  as follows. Let us say that a term of the form (11.12) has order  $|\gamma|$ . We note that the claim at order 1 is an immediate consequence of the multivariate chain rule. Furthermore, this rule also implies that if we apply a partial derivative  $\partial_{x_q}$  to a term of the form (11.12), then we get a sum of terms of the form (11.12) of the next order. Thus, if the claim is true at a given order, then it is true at the next order. It then follows that it holds at any order. This proves the claim.  $\square$

Thanks to the claim above we know that in (11.11) each partial derivative  $\partial_x^\gamma(b_{lk} \circ \phi)(0)$  is a linear combination of terms of the form,

$$(11.13) \quad \partial_x^{\gamma_1} \phi_{q_1}(0) \cdots \partial_x^{\gamma_p} \phi_{q_p}(0) (\partial_{x'_{q_1}} \cdots \partial_{x'_{q_p}} b_{lk}^{X'})(0),$$

where  $p, q_1, \dots, q_p, \gamma_1, \dots, \gamma_p$  are as in (11.12). We observe that

$$\begin{aligned} (w'_{q_1} - \langle \gamma_1 \rangle) + \cdots + (w'_{q_p} - \langle \gamma_p \rangle) + (w'_k - w'_l - (w'_{q_1} + \cdots + w'_{q_p})) \\ = w'_k - w'_l - \langle \gamma \rangle \geq w'_k - w_j - \langle \alpha \rangle > 0. \end{aligned}$$

Therefore, at least one of the numbers  $w'_{q_s} - \langle \gamma_s \rangle$ ,  $s = 1, \dots, p$ , or  $w'_k - w'_l - (w'_{q_1} + \cdots + w'_{q_p})$  must be positive. If  $w'_{q_s} - \langle \gamma_s \rangle > 0$ , then, as  $|\gamma_s| \leq |\gamma| \leq |\alpha| \leq m$ , the induction assumption implies that  $\partial_x^{\gamma_s} \phi_{q_s}(0) = 0$ . If  $w'_k - w'_l - (w'_{q_1} + \cdots + w'_{q_p}) > 0$ , then (11.10) implies that  $(\partial_{x'_{q_1}} \cdots \partial_{x'_{q_p}} b_{lk}^{X'})(0) = 0$ . In any case we see that all the terms (11.13) vanish, and hence  $\partial_x^\gamma(b_{lk}^{X'} \circ \phi)(0) = 0$ . It then follows that the partial derivative of order  $\alpha$  at  $x = 0$  of the r.h.s. of (11.7) is zero. As the partial derivative of order  $\alpha$  at  $x = 0$  of the l.h.s. agrees with  $\partial_x^\alpha \partial_{x_j} \phi_k(0)$ , we deduce that

$$\partial_x^\alpha \partial_{x_j} \phi_k(0) = 0 \quad \text{whenever } w_k - \langle \alpha \rangle - w_j > 0 \text{ and } |\alpha| \leq m.$$

This proves that (11.8) holds up to order  $m + 1$ . The proof is complete.  $\square$

As the component  $\phi_k(x)$ ,  $k = 1, \dots, n'$ , has weight  $w'_k$ , using Lemma 5.4 we may write

$$\phi_k(x) = \hat{\phi}_k(x) + \Theta_k(x),$$

where  $\hat{\phi}_k(x) = \sum_{\langle \alpha \rangle = w'_k} (\alpha!)^{-1} \partial_x^\alpha \phi_k(0) x^\alpha$  and  $\Theta_k(x)$  has weight  $\geq w'_k + 1$ . Then, for all  $x \in \mathbb{R}^n$ , we have

$$t^{-w'_k} \cdot \phi_k(t \cdot x) = \hat{\phi}_k(x) + O(t) \quad \text{as } t \rightarrow 0^+.$$

Therefore, setting  $\hat{\phi}(x) = (\hat{\phi}_1(x), \dots, \hat{\phi}_{n'}(x))$ , we see that, for all  $x \in \mathbb{R}^n$ , we have

$$t^{-1} \cdot \phi(t \cdot x) = \hat{\phi}(x) + O(t) \quad \text{as } t \rightarrow 0^+.$$

Combining this with Lemma 8.3 we then deduce that

$$(11.14) \quad \phi(x) = \hat{\phi}(x) + O(\|x\|^{w+1}) \quad \text{near } x = 0.$$

**Lemma 11.4.** *The maps  $\hat{\phi}$  and  $\phi'_H(a)$  agree.*

*Proof.* Let  $(\phi'_H(a)_{jk})$  be the matrix of  $\phi'_H(a)$ . As  $\hat{\phi}(0) = 0$  and  $\phi'_H(a)$  is a linear map, we only have to prove that

$$(11.15) \quad \partial_{x_j} \hat{\phi}_k(x) = \phi'_{H(a)}(a)_{kj} \quad \text{for } j = 1, \dots, n \text{ and } k = 1, \dots, n'.$$

We note that  $\hat{\phi}_k$  is a polynomial map homogeneous of degree  $w'_k$ , and so it does not depend on the variables  $x_j$  with  $w_j > w'_k$ . Thus, in this case we have

$$\partial_{x_j} \hat{\phi}_k(x) = 0 = \phi'_{H(a)}(a)_{jk}.$$

This shows that (11.15) holds when  $w'_k - w_j < 0$ .

Bearing this in mind, we note that in view of (2.23) the vector fields  $X_j^a$ ,  $j = 1, \dots, n$ , and  $(X'_k)^{a'}$ ,  $k = 1, \dots, n'$ , are of the form,

$$X_j^a = \sum_{w_l \geq w_j} b_{jl}^a(x) \partial_{x_l} \quad \text{and} \quad (X'_k)^{a'} = \sum_{w_{p'} \geq w'_k} b_{kp'}^{a'}(x') \partial_{x'_{p'}},$$

where  $b_{jl}^a(x)$  and  $b_{kp}^{a'}(x')$  are polynomials of the form,

$$(11.16) \quad b_{jl}^a(x) = \delta_{jl} + \sum_{\substack{w_l - \langle \alpha \rangle = w_j \\ \alpha \neq 0}} b_{jl\alpha}^a x^\alpha \quad \text{and} \quad b_{kp}^{a'}(x') = \delta_{kp} + \sum_{\substack{w'_p - \langle \beta \rangle = w'_k \\ \beta \neq 0}} b_{kp\beta}^{a'} (x')^\beta,$$

where  $b_{jl\alpha}^a$  and  $b_{kp\beta}^{a'}$  are some real constants. In terms of these coefficients, the equality (11.4) implies that

$$(11.17) \quad \phi'_H(a)_{kj} + \sum_{\substack{w_j < w_l \leq w'_k \\ \langle \alpha \rangle = w_l - w_j}} b_{jl\alpha}^a x^\alpha \phi'_H(a)_{kl} = \sum_{w'_l = w_j} c_{jl}(a) b_{lk}^{a'} (\phi'_H(a)x) \quad \text{for } w_j \leq w'_k,$$

where we have used the fact that  $b_{jl}^a(x) = \delta_{jl}$  for  $w_l \leq w_j$ .

As it turns out, the bulk of the proof is establishing a similar formula for  $\hat{\phi}(x)$ .

*Claim.* For  $w_j \leq w'_k$ , we have

$$(11.18) \quad \partial_{x_j} \hat{\phi}_k(x) + \sum_{\substack{w_j < w_l \leq w'_k \\ \langle \alpha \rangle = w_l - w_j}} b_{jl\alpha}^a x^\alpha \partial_{x_l} \hat{\phi}_k(x) = \sum_{w'_l = w_j} c_{jl}(a) b_{lk}^{a'} (\hat{\phi}(x)).$$

*Proof of the Claim.* Let  $U$  be the range of the local coordinates  $(x_1, \dots, x_n)$  and  $U'$  the range of the local coordinates  $(x'_1, \dots, x'_{n'})$ . As  $(x_1, \dots, x_n)$  are Carnot coordinates at  $a$  adapted to  $(X_1, \dots, X_n)$ , Proposition 7.2 ensures us that, for  $j = 1, \dots, n$  and as  $t \rightarrow 0^+$ , we have

$$(11.19) \quad t^{w_j} \delta_t^* X_j = X_j^a + O(t) \quad \text{in } \mathcal{X}(U).$$

In terms of the coefficients  $b_{jl}^X(x)$  this implies that, as  $t \rightarrow 0^+$ , we have

$$(11.20) \quad t^{w_j - w_l} b_{jl}^X(t \cdot x) = b_{jl}^a(x) + O(t) \quad \text{in } C^\infty(U).$$

Likewise, for  $k, p = 1, \dots, n'$  and as  $t \rightarrow 0^+$ , we have

$$(11.21) \quad t^{w'_k - w'_p} b_{kp}^{X'}(x') = b_{kp}^{a'}(x') + O(t) \quad \text{in } C^\infty(U').$$

We also note that (11.14) and Lemma 8.3 imply that, as  $t \rightarrow 0^+$ , we have

$$(11.22) \quad t^{-1} \cdot \phi(t \cdot x) = \hat{\phi}(x) + O(t) \quad \text{in } C^\infty(U, \mathbb{R}^{n'}).$$

In particular, we may termwise differentiate the asymptotics to see that, for  $j = 1, \dots, n$  and  $k = 1, \dots, n'$ , and for all  $x \in \mathbb{R}^n$ , we have

$$t^{w_j - w'_k} \partial_{x_j} \phi_k(t \cdot x) = \partial_{x_j} \hat{\phi}_k(x) + O(t) \quad \text{as } t \rightarrow 0^+.$$

Bearing all this mind, substituting  $t \cdot x$  for  $x$  in (11.7) and rescaling both sides of the equation by  $t^{w_j - w'_k}$  gives

$$(11.23) \quad \sum_{1 \leq l \leq n} t^{w_j - w_l} b_{jl}^X(t \cdot x) t^{w_l - w'_k} \partial_{x_l} \phi_k(t \cdot x) = \sum_{\substack{1 \leq l \leq n' \\ w'_l \leq w_j}} t^{w_j - w'_l} c_{jl}(t \cdot x) t^{w'_l - w'_k} b_{lk}^{X'}(t \cdot (t^{-1} \cdot \phi(t \cdot x))).$$

Assume  $w_j \leq w'_k$ . Then letting  $t \rightarrow 0^+$  in (11.23) and using (11.19)–(11) we obtain

$$\sum_{w_j \leq w_l \leq w'_k} b_{jl}^a(x) \partial_{x_l} \hat{\phi}_k(x) = \sum_{w'_l = w_j} c_{jl}(a) b_{lk}^{a'}(\hat{\phi}(x)).$$

Combining this with the fact that  $b_{jl}^a(x) = \delta_{jl}$  for  $w_l \leq w_j$  gives (11.18) and proves the claim.  $\square$

We shall now proceed to prove (11.15) by induction. As mentioned above this equality holds when  $w'_k - w_j < 0$ . Given  $m \in \mathbb{N}_0$ , assume that (11.15) holds for  $w'_k - w_j < m$ . We note that if  $w'_k \leq m$ , then, for all  $j = 1, \dots, n$ , we have  $w'_k - w_j \leq m - 1 < m$ , and so  $\partial_{x_j} \hat{\phi}_k(x) = \phi'_H(a)_{kj}$ . As  $\hat{\phi}(0) = 0$ , we then deduce that

$$(11.24) \quad \hat{\phi}_k(x) = \sum_{1 \leq j \leq n} \phi'_H(a)_{kj} x_j = \phi'_H(a)_k x \quad \text{for } w'_k \leq m.$$

Let  $j \in \{1, \dots, n\}$  and  $k \in \{1, \dots, n'\}$  be such that  $w'_k - w_j \leq m$ . We observe that if  $w_j < w_l \leq w'_k$ , then  $w'_k - w_l < m$ , and so  $\partial_{x_l} \hat{\phi}_k(x) = \phi'_H(a)_{kl}$ . Therefore, we see that when  $w'_k - w_j \leq m$  the left-hand side of (11.18) is equal to

$$\partial_{x_j} \hat{\phi}_k(x) + \sum_{\substack{w_j < w_l \leq w'_k \\ \langle \alpha \rangle = w_l - w_j}} b_{jl\alpha}^a x^\alpha \phi'_H(a)_{kl}.$$

Comparing this to (11.17) then shows that when  $w'_k - w_j \leq m$  we have

$$(11.25) \quad \partial_{x_j} \hat{\phi}_k(x) - \phi'_H(a)_{kj} = \text{l.h.s. of (11.18)} - \text{l.h.s. of (11.17)}.$$

Bearing this in mind, we observe that (11.16) shows that  $b_{lk}^{a'}(x')$  is a linear combination of monomials  $x^\beta$  with  $\langle \alpha \rangle = w'_k - w'_l$ . Therefore, it may only involve variables  $x'_p$  with  $w'_p \leq w'_k - w'_l$ . Thus, if  $w'_k - w'_l \leq m$ , then  $b_{lk}^{a'}(x')$  only depends on the variables  $x'_p$  with  $w'_p \leq m$ . As (11.24) ensures that  $\hat{\phi}_p(x) = \phi'_H(a)_p x$  when  $w'_p \leq m$ , we deduce that

$$b_{lk}^{a'}(\hat{\phi}(x)) = b_{lk}^{a'}(\phi'_H(a)x) \quad \text{for } w'_k - w'_l \leq m.$$

It then follows that when  $w'_k - w_j \leq m$  the right-hand sides of (11.18) and (11.17) agree. Combining this with (11.25) we obtain

$$\partial_{x_j} \hat{\phi}_k(x) - \phi'_H(a)_{kj} = 0 \quad \text{when } w'_k - w_j \leq m.$$

This shows that if (11.15) holds for  $w'_k - w_j < m$ , then it must hold for  $w'_k - w_j \leq m$ . It then follows that it holds for all  $j = 1, \dots, n$  and  $k = 1, \dots, n'$ . As mentioned above, this proves that  $\hat{\phi}(x) = \phi'_H(a)x$ . The proof is complete.  $\square$

Combining (11.14) and Lemma 11.4 we then obtain the following approximation result.

**Theorem 11.5.** *Let  $(x_1, \dots, x_n)$  be Carnot coordinates at  $a$  adapted to the  $H$ -frame  $(X_1, \dots, X_n)$  and  $(x_1, \dots, x_{n'})$  Carnot coordinates at  $a'$  adapted to the  $H'$ -frame  $(X_1, \dots, X_{n'})$ . Then, in these coordinates, we have*

$$\phi(x) = \phi'_H(a)x + O_w(\|x\|^{w+1}) \quad \text{near } x = 0.$$

*Remark 11.6.* Bellaïche [Be, Prop. 7.29] also obtained an approximation result of the form (11.14) in privileged coordinates when  $\phi$  is a Carnot diffeomorphism. However, it did not identify the map  $\hat{\phi}$  with  $\phi'_H(a)$ . In fact, if we use general privileged coordinates, the map  $\hat{\phi}$  in (11.14) is only a Lie algebra map between the Lie algebras of the extrinsic groups  $G^{(a)}$  and  $G^{(a')}$  (see [Be, loc. cit.]). It is only by working in Carnot coordinates that we get the identification with  $\phi'_H(a)$  and obtain the approximation by a Lie group map.

*Remark 11.7.* In the case of Heisenberg manifolds a version of Theorem 11.5 was obtained for Heisenberg diffeomorphisms in [Po1].

*Remark 11.8.* Theorem 11.5 shows that the *algebraic* tangent map  $\phi'_H(a)$  constructed in Section 2 actually provides us with a *differential-geometric* notion of tangent map. This result has numerous consequences, in particular this result is paramount to the construction of the tangent groupoid in Section 13. Moreover, as mentioned in the introduction, it should play an important role in the construction of a full hypoelliptic pseudodifferential calculus on Carnot calculus.

In the rest of the section, we shall mention a few immediate consequences of Theorem 11.5. A first consequence is a description of the behavior of Carnot coordinates under the action of Carnot diffeomorphisms. Namely, we have the following result.

**Proposition 11.9.** *Assume that  $\phi$  is a Carnot diffeomorphism. Let  $(x_1, \dots, x_n)$  be Carnot coordinates at  $a$  adapted to  $(X_1, \dots, X_n)$  and  $(x'_1, \dots, x'_{n'})$  local coordinates near  $a'$ . Then the change of variables  $x' \rightarrow \phi'_H(a) \circ \phi^{-1}(x')$  provides us with Carnot coordinates at  $a'$  adapted to  $(X'_1, \dots, X'_{n'})$ .*

*Proof.* Let  $\varepsilon_{a'}^{X'}$  be the normal Carnot coordinate map associated with  $(X'_1, \dots, X'_n)$  and the local coordinates  $(x'_1, \dots, x'_n)$ . We note that  $\varepsilon_a^X \circ \phi$  is the map  $\phi$  in the Carnot coordinates  $(x_1, \dots, x_n)$  and the Carnot coordinates provided by  $\varepsilon_a^{X'}$ . Therefore, by Theorem 11.5, near  $x = 0$ , we have

$$\varepsilon_{a'}^{X'} \circ \phi(x) = \phi'_H(a)x + O_w(\|x\|^{w+1}).$$

Using Lemma 8.3 we deduce that, for all  $x \in \mathbb{R}^n$ , we have

$$t^{-1} \cdot \varepsilon_{a'}^{X'} \circ \phi(x) = \phi'_H(a)x + O(t) \quad \text{as } t \rightarrow 0^+.$$

If we substitute  $\phi'_H(a)(x)$  for  $x$  and use the  $w$ -homogeneity of  $\phi'_H(a)$ , then we see that, as  $t \rightarrow 0^+$ , we have

$$t^{-1} \cdot \varepsilon_{a'}^{X'} \circ \phi \circ (\phi'_H(a))^{-1}(t \cdot x) = t^{-1} \cdot \varepsilon_{a'}^{X'} \circ \phi(t \cdot (\phi'_H(a))^{-1}x) = x + O(t).$$

In view of Lemma 8.3 and Lemma 8.6 this shows that, near  $x = a$ , we have

$$\varepsilon_{a'}^{X'}(x) = \phi'_H(a) \circ \phi^{-1}(x) + O_w(\|\phi \circ (\phi'_H(a))^{-1}(x)\|^{w+1}).$$

Using Lemma 8.7 we deduce that, near  $x = a$ , we have

$$\phi'_H(a) \circ \phi^{-1}(x) = \varepsilon_{a'}^{X'}(x) + O_w(\|\varepsilon_{a'}^{X'}(x)\|^{w+1}).$$

Combining this with Theorem 9.1 then proves that the change of variables  $x' \rightarrow \phi'_H(a) \circ \phi^{-1}(x')$  provides us with Carnot coordinates at  $a'$  adapted to  $(X'_1, \dots, X'_n)$ . The proof is complete.  $\square$

Let us now interpret Theorem 11.5 in terms of normal Carnot coordinates.

**Proposition 11.10.** *Let  $(x_1, \dots, x_n)$  be local coordinates near  $a$  and  $(x'_1, \dots, x'_{n'})$  local coordinates near  $a'$ .*

(1) *Near  $x = a$ , we have*

$$(11.26) \quad \varepsilon_{a'}^{X'} \circ \phi(x) = \phi'_H(a) \circ \varepsilon_a^X(x) + O_w(\|\varepsilon_a^X(x)\|^{w+1}).$$

(2) *Assume further that  $\phi$  is a Carnot diffeomorphism. Then*

$$(11.27) \quad \varepsilon_{\phi(a)}^{\phi_* X} \circ \phi(x) = \varepsilon_a^X(x) + O_w(\|\varepsilon_a^X(x)\|^{w+1}) \quad \text{near } x = a.$$

*Proof.* As  $\varepsilon_{a'}^{X'} \circ \phi \circ (\varepsilon_a^X)^{-1}$  is the map  $\phi$  in the Carnot coordinates provided by  $\varepsilon_a^X$ , Theorem 11.5 ensures us that

$$\varepsilon_{a'}^{X'} \circ \phi \circ (\varepsilon_a^X)^{-1}(x) = \phi'_H(a)x + O(\|x\|^{w+1}) \quad \text{near } x = 0.$$

Combining this with Lemma 8.6 gives the asymptotics (11.26). If we further assume that  $\phi$  is a Carnot diffeomorphism and  $X'_j = \phi_* X_j$  for  $j = 1, \dots, n$ , then thanks to the 3rd part of Lemma 11.1 we know that  $\phi'_H(a)$  is the identity map. Therefore, in this case, the asymptotics (11.26) reduces to (11.27). The proof is complete.  $\square$

*Remark 11.11.* Proposition 10.1 shows that the normal Carnot coordinate map *a priori* depends non-trivially on the partial derivatives of the coefficients of the  $H$ -frame  $X = (X_1, \dots, X_n)$ . Therefore, it is not functorial with respect to Carnot diffeomorphisms. Nevertheless, the asymptotics (11.27) shows that some form of *asymptotic* functoriality actually holds.

As it turns out, we actually have a stronger statement than the asymptotics (11.26).

**Proposition 11.12.** *Let  $(x_1, \dots, x_n)$  be local coordinates near  $a$  and  $(x'_1, \dots, x'_{n'})$  local coordinates near  $a'$ . Let us denote by  $U$  the range of the coordinates  $(x_1, \dots, x_n)$  and set  $\mathcal{U} = \{(y, x, t) \in U \times \mathbb{R}^n \times [0, \infty); \varepsilon_y^{-1}(t \cdot x) \in U\}$ . Then there is a smooth map  $\Theta : \mathcal{U} \rightarrow \mathbb{R}^{n'}$  such that*

$$t^{-1} \cdot \left[ \varepsilon_{\phi(y)}^{X'} \circ \phi \circ (\varepsilon_y^X)^{-1}(t \cdot x) \right] = \phi'_H(y)x + t\Theta(y, x, t) \quad \forall (y, x, t) \in \mathcal{U}.$$



*Proof.* It follows from Proposition 10.3 that  $(y, x) \rightarrow \varepsilon_y^{-1}(x)$  is a smooth map from  $U \times \mathbb{R}^n$  to  $\mathbb{R}^n$ , and so  $\mathcal{U}$  is an open subset of  $U \times \mathbb{R}^n \times [0, \infty)$ . For  $(y, x, t) \in \mathcal{U}$  set

$$\tilde{\Theta}(y, x, t) = \left[ \varepsilon_{\phi(y)}^{X'} \circ \phi \circ (\varepsilon_y^X)^{-1}(t \cdot x) \right] - t \cdot [\phi'_H(y)x].$$

It follows from (11.3) that the map  $(y, x) \rightarrow \phi'_H(y)x$  is smooth from  $U \times \mathbb{R}^n$  to  $\mathbb{R}^{n'}$ . Combining this with Proposition 10.3 we then deduce that  $\tilde{\Theta}(y, x, t)$  is a smooth map from  $\mathcal{U}$  to  $\mathbb{R}^{n'}$ . Moreover, by Proposition 11.10 we know that, for all  $(y, x) \in U \times \mathbb{R}^n$ , we have

$$t^{-1} \cdot \tilde{\Theta}(y, x, t) = O(t) \quad \text{as } t \rightarrow 0^+.$$

Therefore, using Lemma 8.1 we conclude there is smooth map  $\Theta : \mathcal{U} \rightarrow \mathbb{R}^{n'}$  such that, for all  $(y, x, t) \in \mathcal{U}$ , we have

$$t\Theta(y, x, t) = t^{-1} \cdot \tilde{\Theta}(y, x, t) = t^{-1} \cdot \left[ \varepsilon_{\phi(y)}^{X'} \circ \phi \circ (\varepsilon_y^X)^{-1}(t \cdot x) \right] - \phi'_H(y)x.$$

This proves the result.  $\square$

*Remark 11.13.* Proposition 11.12 will be an important tool in the construction of the tangent groupoid of a Carnot manifold in Section 13. In particular, it implies that the asymptotics (11.26)–(11.27) hold locally uniformly with respect to  $a$ .

*Remark 11.14.* The key ingredients in the proof of Proposition 11.12 are the smoothness of the map  $(y, x) \rightarrow \varepsilon_y(x)$  and the asymptotics (11.26), which is a consequence of Theorem 11.5. Therefore, Proposition 11.12 remains valid if we replace  $\varepsilon_y^X$  and  $\varepsilon_{y'}^{X'}$  by smooth fields of Carnot coordinate maps  $y \rightarrow \phi_y$  and  $y' \rightarrow \phi_{y'}$ .

*Remark 11.15.* If we use Proposition 10.1, then a slight elaboration of the proof of Proposition 11.12 shows that the map  $\Theta$  depends continuously on the diffeomorphism  $\phi$  and the vector fields  $X_j$ ,  $j = 1, \dots, n$ , and  $X'_k$ ,  $k = 1, \dots, n'$ .

Finally, in Section 10 it was shown that the normal Carnot coordinate map  $\varepsilon_a^X$  depends in a non-trivial way on the coefficients of the  $H$ -frame  $(X_1, \dots, X_n)$ . However, as the following shows, the normal Carnot coordinate map transforms very nicely under rescalings of the  $H$ -frame.

**Proposition 11.16.** *Let  $(x_1, \dots, x_n)$  be local coordinates near  $a$  and denote by  $U \subset \mathbb{R}^n$  their range. Let  $t > 0$  and  $y \in U$  be such that  $t \cdot y \in U$ . In addition, for  $j = 1, \dots, n$ , set  $\hat{X}_j = t^{w_j} \delta_t^* X_j$ . Then*

$$t^{-1} \cdot \varepsilon_{t \cdot y}^X(t \cdot x) = \varepsilon_y^{\hat{X}}(x) \quad \text{for all } x \in \mathbb{R}^n,$$

where  $\varepsilon_y^{\hat{X}}$  is the normal Carnot coordinate map associated with  $y$  and the frame  $(\hat{X}_1, \dots, \hat{X}_n)$ .

*Proof.* Applying Proposition 11.10 to  $\phi = \delta_t$  and  $a = y$  shows that

$$\varepsilon_{t \cdot y}^X \circ \delta_t(x) = (\delta_t)'_H(y) \circ \varepsilon_y^{\hat{X}}(x) + O_w\left(\|\varepsilon_y^{\hat{X}}(x)\|\right) \quad \text{near } x = y.$$

As  $(\delta_t)'_H(y) = \delta_t$  and  $\delta_t$  is  $w$ -homogeneous, we get

$$\delta_t^{-1} \circ \varepsilon_{t \cdot y}^X \circ \delta_t(x) = \varepsilon_y^{\hat{X}}(x) + O_w\left(\|\varepsilon_y^{\hat{X}}(x)\|\right).$$

By Theorem 9.1 this implies that the map  $\delta_t^{-1} \circ \varepsilon_{t \cdot y}^X \circ \delta_t$  provides us with Carnot coordinates at  $y$  adapted to  $(\hat{X}_1, \dots, \hat{X}_n)$ .

Let us write  $\varepsilon_{t \cdot y}^X = \hat{\varepsilon} \circ T$ , where  $\hat{\varepsilon}$  is a  $w$ -homogeneous polynomial diffeomorphism of the form (9.7) and  $T$  is an affine map such that  $T(t \cdot y) = 0$ . Then

$$\delta_t^{-1} \circ \varepsilon_{t \cdot y}^X \circ \delta_t = (\delta_t^{-1} \circ \hat{\varepsilon} \circ \delta_t) \circ (\delta_t^{-1} \circ T \circ \delta_t).$$

We observe that  $\delta_t^{-1} \circ \hat{\varepsilon} \circ \delta_t$  is of the form (9.7) and  $\delta_t^{-1} \circ T \circ \delta_t$  is an affine map such that  $(\delta_t^{-1} \circ T \circ \delta_t)(y) = t^{-1} \cdot T(t \cdot y) = 0$ . Using Theorem 9.2 we then deduce that  $\delta_t^{-1} \circ \varepsilon_{t \cdot y}^X \circ \delta_t = \varepsilon_y^{\hat{X}}$ . This proves the result.  $\square$

## 12. SMOOTH DEFORMATION TO THE PRODUCT OF $GM$

In this section, we shall relate the normal Carnot coordinate map to the product of  $GM$  and show that this provides us with a smooth deformation to the product of  $GM$ .

In what follows, given  $a \in M$ , we let  $(X_1, \dots, X_n)$  be an  $H$ -frame near  $a$ . When working in local coordinates we shall tacitly assume that the vector fields  $X_j$ ,  $j = 1, \dots, n$ , are defined on the domain of these local coordinates and we will identify them with the corresponding vector fields in the local coordinates. In addition, as above, we shall identify  $GM(a)$  with the group obtained in the coordinates defined by the basis  $\{\dot{X}_j(a)\}_{1 \leq j \leq n}$  of  $\mathfrak{g}M(a)$ .

**Definition 12.1.** Given  $m \in \mathbb{N}_0$ , let  $\Theta(x, y) = (\Theta_1(x, y), \dots, \Theta_n(x, y))$  be a smooth map from  $U$  to  $\mathbb{R}^n$ , where  $U$  is an open neighborhood of  $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^n$ . We say that  $\Theta(x, y)$  is  $O_w((\|x\| + \|y\|)^{w+m})$ , and write  $\Theta(x, y) = O_w((\|x\| + \|y\|)^{w+m})$ , when, for  $j = 1, \dots, n$ , we have

$$\Theta_j(x, y) = O((\|x\| + \|y\|)^{w_j+m}) \quad \text{near } (x, y) = (0, 0).$$

Along lines similar to that of the proof of Lemma 8.1 we obtain the following statement.

**Lemma 12.2.** Let  $\Theta(x, y) = (\Theta_1(x, y), \dots, \Theta_n(x, y))$  be a smooth map from  $U$  to  $\mathbb{R}^n$ , where  $U$  is an open neighborhood of  $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^n$ . In addition, set

$$\mathcal{U} = \{(x, y, t) \in U \times [0, \infty); (t \cdot x, t \cdot y) \in U\}.$$

Then the following are equivalent:

- (1) The function  $\Theta(x, y)$  is  $O_w((\|x\| + \|y\|)^{w+m})$  near  $(x, y) = (0, 0)$ .
- (2) For all  $x \in \mathbb{R}^n$ , we have

$$t^{-1} \cdot \Theta(t \cdot x, t \cdot y) = O(t^m) \quad \text{as } t \rightarrow 0^+.$$

- (3) As  $t \rightarrow 0^+$ , we have

$$t^{-1} \cdot \Theta(t \cdot x, t \cdot y) = O(t^m) \quad \text{in } C^\infty(U \times U, \mathbb{R}^n).$$

- (4) There is a smooth function on  $\tilde{\Theta}(x, y, t)$  from  $\mathcal{U}$  to  $\mathbb{R}^n$  such that

$$t^{-1} \cdot \Theta(t \cdot x, t \cdot y) = t \tilde{\Theta}(x, y, t) \quad \text{for all } (x, t) \in \mathcal{U}.$$

We are now in a position to state the main result of this section.

**Proposition 12.3.** Let  $(x_1, \dots, x_n)$  be Carnot coordinates at  $a$  adapted to  $(X_1, \dots, X_n)$ . Then, near  $(x, y) = (0, 0)$ , we have

$$(12.1) \quad \varepsilon_y^X(x) = (-y) \cdot x + O_w((\|x\| + \|y\|)^{w+1}),$$

where  $\cdot$  is the group law of  $GM(a)$  in the coordinates defined by  $\{\dot{X}_j(a)\}$ .

*Proof.* Let us denote by  $U$  the range of the local coordinates  $(x_1, \dots, x_n)$ . In addition, for  $t > 0$  and  $j = 1, \dots, n$ , set  $\hat{X}_j^t = t^{w_j} \delta_t^* X_j$ . Let  $x \in \mathbb{R}^n$  and  $y \in \mathbb{R}^n$  be such that  $t \cdot y \in U$ . Then by Proposition 11.16 we have

$$t^{-1} \cdot \varepsilon_{t \cdot y}^X(t \cdot x) = \varepsilon_y^{\hat{X}^t}(x),$$

where  $\varepsilon_y^{\hat{X}^t}(x)$  is the normal Carnot coordinate map associated with  $(\hat{X}_1^t, \dots, \hat{X}_n^t)$ . Moreover, as  $(x_1, \dots, x_n)$  provides us with Carnot coordinates at  $a$  adapted to  $(X_1, \dots, X_n)$ , we know by Proposition 7.2 that, as  $t \rightarrow 0^+$  and for  $j = 1, \dots, n$ , we have

$$\hat{X}_j^t = t^{w_j} \delta_t^* X_j = X_j^a + O(t) \quad \text{in } \mathcal{X}(U).$$

Combining this with Proposition 10.4 we deduce that, as  $t \rightarrow 0^+$ , we have

$$\varepsilon_y^{\hat{X}^t}(x) = \varepsilon_y^{X^a}(x) + O(t) \quad \text{in } C^\infty(U, \mathbb{R}^n),$$

where  $\varepsilon_y^{X^a}$  is the normal Carnot coordinate map associated with  $y$  and the frame  $(X_1^a, \dots, X_n^a)$ . As by Proposition 9.3 we know that  $\varepsilon_y^{X^a} = (-y) \cdot x$ , it then follows that, for all  $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ , we have

$$t^{-1} \cdot \varepsilon_{t \cdot y}^X(t \cdot x) = \varepsilon_y^{\hat{X}^t}(x) = (-y) \cdot x + O(t) \quad \text{as } t \rightarrow 0^+.$$

Combining this with Lemma 12.2 gives (12.1) and completes the proof.  $\square$

Combining Proposition 12.3 with Lemma 8.3 gives the following statement.

**Proposition 12.4.** *Let  $(x_1, \dots, x_n)$  be Carnot coordinates at  $a$  adapted to  $(X_1, \dots, X_n)$ . Denote by  $U$  their range and set  $\mathcal{U} = \{(y, x, t) \in U \times \mathbb{R}^n \times [0, \infty); (\varepsilon_y^X)^{-1}(t \cdot x) \in U\}$ . Then there is a smooth map  $\Theta : \mathcal{U} \rightarrow \mathbb{R}^n$  such that*

$$t^{-1} \cdot \varepsilon_{t \cdot y}^X(t \cdot x) = (-y) \cdot x + t\Theta(x, y, t) \quad \text{for all } (y, x, t) \in \mathcal{U}.$$

*Remark 12.5.* The above result provides us with a smooth deformation to the product of  $GM(a)$ . This will play an important role in the construction of the tangent groupoid of a Carnot manifold in Section 13.

### 13. THE TANGENT GROUPOID OF A CARNOT MANIFOLD

In this section, we explicitly construct a tangent groupoid for any Carnot manifold  $(M, H)$  as a differentiable groupoid encoding the smooth deformation of  $M \times M$  to  $GM$ . In the case of equiregular Carnot-Carathéodory manifold the existence of such a groupoid was conjectured by Bellaïche [Be]. In particular, this will show that the tangent group bundle  $GM$  is tangent to  $M$  in a differentiable fashion and gives a conceptual explanation on why its fibers must be Carnot type groups.

**13.1. Differentiable groupoids.** Let us recall the formal definition of a groupoid.

**Definition 13.1.** A groupoid structure on a set  $\mathcal{G}$  is given by

- (i) A set  $\mathcal{G}^{(0)}$  (called base) and a one-to-one map  $\epsilon : \mathcal{G}^{(0)} \rightarrow \mathcal{G}$  (called unit map).
- (ii) Maps  $s : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$  and  $r : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$  (called range and source maps, respectively) such that

$$s[\epsilon(\gamma)] = r[\epsilon(\gamma)] = \epsilon(\gamma) \quad \text{for all } \gamma \in \mathcal{G}.$$

- (iii) A multiplication map,

$$\mu : \mathcal{G}^{(2)} \ni (\gamma_1, \gamma_2) \rightarrow \gamma_1 \cdot \gamma_2 \in \mathcal{G}, \quad \text{where } \mathcal{G}^{(2)} = \{(\gamma_1, \gamma_2) \in \mathcal{G}^2; s(\gamma_1) = r(\gamma_2)\},$$

satisfying the following properties:

$$\gamma \cdot \epsilon[s(\gamma)] = \epsilon[r(\gamma)] \cdot \gamma = \gamma \quad \text{for all } \gamma \in \mathcal{G},$$

$$s(\gamma_1 \cdot \gamma_2) = s(\gamma_2) \quad \text{and} \quad r(\gamma_1 \cdot \gamma_2) = r(\gamma_1) \quad \text{for all } (\gamma_1, \gamma_2) \in \mathcal{G}^{(2)},$$

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3 = \gamma_1 \cdot (\gamma_2 \cdot \gamma_3) \quad \text{for all } (\gamma_1, \gamma_2, \gamma_3) \in \mathcal{G}^{(3)},$$

where we have set  $\mathcal{G}^{(3)} = \{(\gamma_1, \gamma_2, \gamma_3) \in \mathcal{G}^3; s(\gamma_1) = r(\gamma_2) \text{ and } s(\gamma_2) = r(\gamma_3)\}$ .

- (iv) An inverse map  $\iota : \mathcal{G} \ni \gamma \rightarrow \gamma^{-1} \in \mathcal{G}$  such that

$$(13.1) \quad \gamma \cdot \gamma^{-1} = \epsilon[r(\gamma)] \quad \text{and} \quad \gamma^{-1} \cdot \gamma = \epsilon[s(\gamma)] \quad \text{for all } \gamma \in \mathcal{G}.$$

*Remark 13.2.* The equations (13.1) uniquely determine the inverse  $\gamma^{-1}$ . This implies that  $(\gamma^{-1})^{-1} = \gamma$  for all  $\gamma \in \mathcal{G}$ , i.e., the inverse map is an involution.

*Remark 13.3.* Given groupoids  $\mathcal{G}$  and  $\mathcal{G}'$ , a groupoid morphism from  $\mathcal{G}$  to  $\mathcal{G}'$  is given by maps  $\Phi : \mathcal{G} \rightarrow \mathcal{G}'$  and  $\Phi^0 : \mathcal{G}^{(0)} \rightarrow (\mathcal{G}')^{(0)}$  such that

$$\begin{aligned} \Phi \circ u &= u \circ \Phi^0, & \Phi^0 \circ s &= s \circ \Phi, & \Phi^0 \circ r &= r \circ \Phi, \\ (\Phi \otimes \Phi) \circ \mu &= \mu \circ \Phi, & \Phi \circ \iota &= \iota \circ \Phi. \end{aligned}$$

*Example 13.4 (Pair Groupoid).* Let  $X$  be a set. Then  $X \times X$  is a groupoid with  $\mathcal{G}^0 = X$ , and the unit, source and range maps defined by

$$\epsilon(x) = (x, x), \quad s(x, y) = x, \quad r(x, y) = y$$

The multiplication and inverse maps are given by

$$(x, y) \cdot (y, z) = (x, z) \quad \text{and} \quad (x, y)^{-1} = (y, x).$$

*Example 13.5 (Group Bundle).* Let  $G \xrightarrow{\pi} M$  be a group bundle over a space  $X$ , so that each fiber  $G(x) = \pi^{-1}(x)$ ,  $x \in X$ , is a group. Then  $G$  defines a groupoid with  $\mathcal{G} = G$  and  $\mathcal{G}^{(0)} = M$ . The unit map is given by

$$u(x) = e_x \quad \text{for all } x \in X,$$

where  $e_x$  is the unit element of  $G(x)$ . The source and range maps are equal to  $\pi$ . This implies that

$$\mathcal{G}^{(2)} = \{(g_1, g_2) \in G^2; \pi(g_1) = \pi(g_2)\}.$$

That is, a pair  $(g_1, g_2) \in G^2$  is in  $\mathcal{G}^{(2)}$  if and only if  $g_1$  and  $g_2$  lie in the same fiber. The multiplication and inverse maps of  $\mathcal{G}$  are given by the fiberwise multiplication and inverse maps of  $G$ .

*Example 13.6 (Group Action).* Let  $X$  be a set equipped with the left-action  $G \times X \ni (g, x) \rightarrow gx \in X$  of some group  $G$ . There is a groupoid associated with this action. We take  $\mathcal{G} = G \times X$  and  $\mathcal{G}^0 = X$ . The unit, source and range maps are defined by

$$\epsilon(g, x) = x, \quad \sigma(g, x) = x, \quad r(g, x) = gx.$$

The multiplication and inverse maps are given by

$$(h, gx) \cdot (g, x) = (hg, x) \quad \text{and} \quad (g, x)^{-1} = (g^{-1}, gx).$$

In what follows, we will be interested in groupoids carrying differentiable structures in the following sense.

**Definition 13.7.** We shall say that a groupoid  $\mathcal{G}$  is  $b$ -differentiable when

- (i)  $\mathcal{G}$  and  $\mathcal{G}^{(0)}$  are manifolds with boundary.
- (ii) The unit map  $\epsilon : \mathcal{G}^{(0)} \rightarrow \mathcal{G}$  is an embedding.
- (iii) The source and range maps  $s : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$  and  $r : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$  are submersions (so that  $\mathcal{G}^{(2)}$  is a manifold with boundary).
- (iv) The multiplication map  $\mu : \mathcal{G}^{(2)} \rightarrow \mathcal{G}$  and the inverse map  $\iota : \mathcal{G} \rightarrow \mathcal{G}$  are smooth maps.

*Remark 13.8.* By smooth maps (resp., embeddings, submersions) between manifolds with boundary we actually mean *boundary preserving* smooth maps (resp., embeddings, submersions).

*Remark 13.9.* When the boundary of  $\mathcal{G}$  is empty we recover the definition of a differentiable groupoid (a.k.a. Lie groupoid). The following are examples of Lie groupoids:

- The pair groupoid associated with a smooth manifold (*cf.* Example 13.4).
- The groupoid associated with a smooth bundle of Lie groups (*cf.* Example 13.5).
- The groupoid associated with the smooth action of a Lie group on a manifold (*cf.* Example 13.6).

An example of  $b$ -differentiable groupoid that is not a Lie groupoid is provided by tangent groupoid of Connes [Co2], the description of which is the purpose of the next subsection.

**13.2. Connes' tangent groupoid.** Given a smooth manifold  $M$ , its *tangent groupoid*  $\mathcal{G} = \mathcal{G}M$  is obtained as follows. At the set-theoretic level we have

$$(13.2) \quad \mathcal{G} = TM \sqcup (M \times M \times (0, \infty)) \quad \text{and} \quad \mathcal{G}^{(0)} = M \times [0, \infty).$$

The unit map  $\epsilon : \mathcal{G}^{(0)} \rightarrow \mathcal{G}$  is defined by

$$\epsilon(x, t) = \begin{cases} (x, x, t) & \text{for } t > 0 \text{ and } x \in M, \\ (x, 0) \in TM & \text{for } t = 0 \text{ and } x \in M. \end{cases}$$

The range and source maps of  $\mathcal{G}$  are given by

$$\begin{aligned} r(x, y, t) &= (x, t) \quad \text{and} \quad s(x, y, t) = (y, t) && \text{for } t > 0 \text{ and } x, y \in M, \\ r(x, v) &= s(x, v) = (x, 0) && \text{for } x \in M \text{ and } v \in T_x M. \end{aligned}$$

We define the multiplication map by setting

$$\begin{aligned} (x, y, t) \cdot (y, z, t) &= (x, y, t) & \text{for } t > 0 \text{ and } x, y, z \in M, \\ (x, v) \cdot (x, w) &= (x, v + w) & \text{for } x \in M \text{ and } v, w \in T_x M. \end{aligned}$$

The inverse map is given by

$$\begin{aligned} (x, y, t)^{-1} &= (y, x, t) & \text{for } t > 0 \text{ and } x, y \in M, \\ (x, v)^{-1} &= (x, -v) & \text{for } x \in M \text{ and } v \in T_x M. \end{aligned}$$

We note that  $\mathcal{G}^{(0)} = M \times [0, \infty)$  is a manifold with boundary and  $\mathcal{G}$  is the disjoint union of the manifolds  $TM$  and  $\mathcal{G}_+ := M \times M \times (0, \infty)$ .

The topology of  $\mathcal{G}$  is the coarsest topology such that

- The unit map  $\epsilon : \mathcal{G}^{(0)} \rightarrow \mathcal{G}$  is continuous.
- The inclusion of  $\mathcal{G}_+ = M \times M \times (0, \infty)$  into  $\mathcal{G}$  is an open continuous map.
- A sequence  $((x_\ell, y_\ell, t_\ell))_{\ell \geq 1} \subset \mathcal{G}_+$  converges to a point  $(x, v) \in TM$  if and only if, as  $\ell \rightarrow \infty$ , we have

$$(13.3) \quad x_\ell \longrightarrow x, \quad y_\ell \longrightarrow x, \quad t_\ell \longrightarrow 0, \quad \text{and}$$

$$(13.4) \quad \frac{1}{t_\ell} (\kappa(y_\ell) - \kappa(x_\ell)) \longrightarrow \kappa'(x)v \quad \text{for any local chart } \kappa \text{ near } x.$$

It can be checked that the condition (13.4) does not depend on the choice of the chart  $\kappa$  near  $x$ .

The differentiable structure of  $\mathcal{G}$  is such that

- The inclusion of  $\mathcal{G}_+ = M \times M \times (0, \infty)$  into  $\mathcal{G}$  is a smooth embedding.
- For any point  $x_0 \in M$  and any local chart  $\kappa$  near  $x_0$ , a local chart from a neighborhood of  $T_{x_0}M$  to  $\mathbb{R}^n \times \mathbb{R}^n \times [0, \infty)$  is given by

$$\begin{aligned} \gamma_\kappa(x, y, t) &= (\kappa(x), t^{-1} (\kappa(y) - \kappa(x)), t) & \text{for } t > 0 \text{ and } x, y \in \text{dom}(\kappa), \\ \gamma_\kappa(x, v) &= (\kappa(x), \kappa'(x)v) & \text{for } x \in \text{dom}(\kappa) \text{ and } v \in T_x M. \end{aligned}$$

If  $\kappa$  and  $\kappa_1$  are local charts near  $x_0$  and  $\phi = \kappa_1 \circ \kappa^{-1}$  is the corresponding transition map, then, for all  $(x, v, t) \in \mathbb{R}^n \times \mathbb{R}^n \times [0, \infty)$  such that  $x$  and  $x + tv$  are in  $\text{dom}(\phi)$ , we have

$$\gamma_{\kappa_1} \circ \gamma_\kappa^{-1}(x, v, t) = \begin{cases} (\phi(x), t^{-1} (\phi(x + tv) - \phi(x)), t) & \text{if } t > 0, \\ (\phi(x), \phi'(x)v, 0) & \text{if } t = 0. \end{cases}$$

Therefore, the maps  $\gamma_\kappa$  do define a system of local charts near  $TM$ . We then have the following result.

**Proposition 13.10** (Connes [Co2]). *With respect to the differentiable structure above, the tangent groupoid  $\mathcal{G} = \mathcal{G}M$  is a  $b$ -differentiable groupoid.*

**13.3. The tangent groupoid of a Carnot manifold.** Let  $(M, H)$  be a Carnot manifold. We associate with it a groupoid  $\mathcal{G} = \mathcal{G}_H M$  as follows. We set

$$\mathcal{G} = GM \sqcup (M \times M \times (0, \infty)) \quad \text{and} \quad \mathcal{G}^{(0)} = M \times [0, \infty).$$

The unit map  $\epsilon : \mathcal{G}^{(0)} \rightarrow \mathcal{G}$  is defined by

$$\epsilon(x, t) = \begin{cases} (x, x, t) & \text{for } t > 0 \text{ and } x \in M, \\ (x, 0) \in GM & \text{for } t = 0 \text{ and } x \in M. \end{cases}$$

The range and source maps of  $\mathcal{G}$  are given by

$$\begin{aligned} r(x, y, t) &= (x, t) \quad \text{and} \quad s(x, y, t) = (y, t) & \text{for } t > 0 \text{ and } x, y \in M, \\ r(x, \xi) &= s(x, \xi) = (x, 0) & \text{for } x \in M \text{ and } \xi \in G_x M. \end{aligned}$$

We define the multiplication map by setting

$$\begin{aligned} (x, y, t) \cdot (y, z, t) &= (x, y, t) & \text{for } t > 0 \text{ and } x, y, z \in M, \\ (x, \xi) \cdot (x, \eta) &= (x, \xi \cdot \eta) & \text{for } x \in M \text{ and } \xi, \eta \in G_x M, \end{aligned}$$

where  $\cdot$  is the product law of  $G_x M$ . The inverse map is given by

$$\begin{aligned} (x, y, t)^{-1} &= (y, x, t) & \text{for } t > 0 \text{ and } x, y \in M, \\ (x, \xi)^{-1} &= (x, \xi^{-1}) = (x, -\xi) & \text{for } x \in M \text{ and } \xi \in G_x M. \end{aligned}$$

It is immediate to check that this defines a groupoid.

**Definition 13.11.**  $\mathcal{G}_H M = \mathcal{G}$  is called the tangent groupoid of the Carnot manifold  $(M, H)$ .

We note that, as in the case of Connes' tangent groupoid,  $\mathcal{G}^{(0)} = M \times [0, \infty)$  is a manifold with boundary and  $\mathcal{G}$  is the disjoint union of the manifolds  $GM$  and  $\mathcal{G}_+ = M \times M \times (0, \infty)$ . We shall now equip  $\mathcal{G}$  with a differentiable structure as follows.

In what follows, given a local  $H$ -chart (cf. Definition 2.14) with associated  $H$ -frame  $(X_1, \dots, X_n)$ , we shall denote by  $(y, x) \rightarrow \varepsilon_y^\kappa(x)$  the normal Carnot coordinate map associated with the local coordinates defined by  $\kappa$  and the frame  $(\kappa_* X_1, \dots, \kappa_* X_n)$ . We often will identify the  $H$ -frame  $(X_1, \dots, X_n)$  with its image under  $\kappa$ .

**Definition 13.12.** The topology of  $\mathcal{G}$  is the coarsest topology such that

- The unit map  $\epsilon : \mathcal{G}^{(0)} \rightarrow \mathcal{G}$  is continuous.
- The inclusion of  $\mathcal{G}_+ = M \times M \times (0, \infty)$  into  $\mathcal{G}$  is an open continuous map.
- A sequence  $((x_\ell, y_\ell, t_\ell))_{\ell \geq 1} \subset \mathcal{G}_+$  converges to a point  $(x, \xi) \in GM$  if and only if, as  $\ell \rightarrow \infty$ , we have

$$(13.5) \quad x_\ell \longrightarrow x, \quad y_\ell \longrightarrow x, \quad t_\ell \longrightarrow 0, \quad \text{and}$$

$$(13.6) \quad t_\ell^{-1} \cdot \left( \varepsilon_{\kappa(x_\ell)}^\kappa \circ \kappa \right) (y_\ell) \longrightarrow \kappa'_H(x) \xi \quad \text{for any local } H\text{-chart } \kappa \text{ near } x.$$

The consistency of (13.6) is the content of the following result.

**Lemma 13.13.** *The condition (13.6) does not depend on the choice of the local  $H$ -chart  $\kappa$ .*

*Proof.* Let  $((x_\ell, y_\ell, t_\ell))_{\ell \geq 1}$  be sequence in  $\mathcal{G}_+$  and  $(x_0, \xi_0)$  a point of  $GM$  such that, as  $\ell \rightarrow \infty$ , we have

$$\begin{aligned} x_\ell &\longrightarrow x_0, & y_\ell &\longrightarrow x_0, & t_\ell &\longrightarrow 0, & \text{and} \\ t_\ell^{-1} \cdot \left( \varepsilon_{\kappa(x_\ell)}^\kappa \circ \kappa \right) (y_\ell) &\longrightarrow \kappa'_H(x_0) \xi_0 & \text{for some local } H\text{-chart } \kappa \text{ near } x_0. \end{aligned}$$

Let  $\kappa_1$  be another local  $H$ -chart near  $x_0$ . Let  $\phi = \kappa_1 \circ \kappa^{-1}$  be the corresponding transition map with domain  $U := \text{ran}(\kappa) \cap \kappa(\text{dom}(\kappa_1))$ . In addition, let  $(X_1, \dots, X_n)$  (resp.,  $(X'_1, \dots, X'_n)$ ) be the  $H$ -frame associated with  $\kappa$  (resp.,  $\kappa_1$ ) in the local coordinates defined by  $\kappa$  (resp.,  $\kappa_1$ ). Note that for  $\ell$  large enough both  $\kappa(x_\ell)$  and  $\kappa(y_\ell)$  are in  $U$ . Setting  $\hat{x}_\ell = \kappa(x_\ell)$  and  $\hat{y}_\ell = t_\ell^{-1} \cdot \left( \varepsilon_{\kappa(x_\ell)}^\kappa \circ \kappa \right) (y_\ell)$ , we then have

$$(13.7) \quad (\varepsilon_{\kappa_1(x_\ell)}^{\kappa_1} \circ \kappa_1)(y_\ell) = \varepsilon_{\hat{x}_\ell}^X \circ \phi \circ \kappa(y_\ell) = \varepsilon_{\hat{x}_\ell}^X \circ \phi \circ (\varepsilon_{\hat{x}_\ell}^X)^{-1} (t_\ell \cdot \hat{y}_\ell).$$

Set  $\mathcal{U} = \{(x, y, t) \in U \times \mathbb{R}^n \times [0, \infty); \varepsilon_x^{-1}(t \cdot y) \in U\}$ . By Proposition 11.12 there is a smooth map  $\Theta : \mathcal{U} \rightarrow \mathbb{R}^n$  such that

$$t^{-1} \cdot \left[ \varepsilon_{\phi(x)}^{X'} \circ \phi \circ (\varepsilon_x^X)^{-1} (t \cdot y) \right] = \phi'_H(x) y + t \Theta(x, y, t) \quad \forall (x, y, t) \in \mathcal{U}.$$

Combining this with (13.7) we see that, for  $\ell$  large enough, we have

$$t_\ell^{-1} \cdot \left[ (\varepsilon_{\kappa_1(x_\ell)}^{\kappa_1} \circ \kappa_1)(y_\ell) \right] = \phi'_H(\hat{x}_\ell) y_\ell + t_\ell \Theta(\hat{x}_\ell, \hat{y}_\ell, t_\ell).$$

We know that  $\hat{x}_\ell \rightarrow \kappa(x_0)$  and  $\hat{y}_\ell \rightarrow \kappa'_H(x_0) \xi_0$  as  $\ell \rightarrow \infty$ . Therefore, as  $\ell \rightarrow \infty$ , we have

$$t_\ell^{-1} \cdot \left[ (\varepsilon_{\kappa_1(x_\ell)}^{\kappa_1} \circ \kappa_1)(y_\ell) \right] = \phi'_H(\kappa(x_0)) (\kappa'_H(x_0) \xi_0) + O(t_\ell) = (\kappa_1)'_H(x_0) \xi_0 + O(t_\ell).$$

This shows that if the condition (13.6) is satisfied by a local  $H$ -chart near  $x_0$ , then it is satisfied by any other local  $H$ -chart near  $x_0$ . The lemma is thus proved.  $\square$

Let  $\kappa$  be a local  $H$ -chart and denote by  $V$  (resp.,  $U$ ) its domain (resp., range). Note that  $V$  (resp.,  $U$ ) is an open subset of  $M$  (resp.,  $\mathbb{R}^n$ ). In addition, we set

$$\begin{aligned}\mathcal{V} &= GM|_V \sqcup (V \times V \times (0, \infty)), \\ \mathcal{U} &= \{(x, y, t) \in U \times \mathbb{R}^n \times [0, \infty); (\varepsilon_x^\kappa)^{-1}(t \cdot y) \in U\}.\end{aligned}$$

We then define the map  $\gamma_\kappa : \mathcal{V} \rightarrow \mathcal{U}$  by

$$(13.8) \quad \begin{aligned}\gamma_\kappa(x, y, t) &= (\kappa(x), t^{-1} \cdot \varepsilon_{\kappa(x)}^\kappa \circ \kappa(y), t) && \text{for } x, y \in V \text{ and } t > 0, \\ \gamma_\kappa(x, \xi) &= (\kappa(x), \kappa'_H(x)\xi) && \text{for } x \in V \text{ and } \xi \in G_x M.\end{aligned}$$

**Lemma 13.14.** *The set  $\mathcal{V}$  is an open subset of  $\mathcal{G}$  and  $\gamma_\kappa$  is a homeomorphism from  $\mathcal{V}$  onto  $\mathcal{U}$  whose inverse map is given by*

$$(13.9) \quad \gamma_\kappa^{-1}(x, y, t) = \begin{cases} (\kappa^{-1}(x), (\varepsilon_x^\kappa \circ \kappa)^{-1}(t \cdot y), t) & \text{for } (x, y, t) \in \mathcal{U} \text{ and } t > 0, \\ (\kappa^{-1}(x), (\kappa^{-1})'_H(x)y) & \text{for } (x, y) \in U \times \mathbb{R}^n \text{ and } t = 0. \end{cases}$$

*Proof.* Let us show that  $\mathcal{V}$  is an open subset of  $\mathcal{G}$  by showing that its complement is a closed subset. We have

$$\mathcal{G} \setminus \mathcal{V} = GM|_{M \setminus V} \sqcup \mathcal{V}_+^c, \quad \text{where } \mathcal{V}_+^c = [(M \times M) \setminus (V \times V)] \times (0, \infty).$$

We observe that  $GM|_{M \setminus V}$  is a closed subset of  $GM$  and  $\mathcal{V}_+^c$  is a closed subset of  $M \times M \times (0, \infty)$ . In addition, let  $((x_\ell, y_\ell, t_\ell))_{\ell \geq 1}$  be a sequence in  $\mathcal{V}_+^c$  that converges in  $\mathcal{G}$  to a point  $(x_0, \xi_0) \in GM$ . This implies that  $x_\ell$  and  $y_\ell$  both converge to  $x_0$  in  $M$ . We observe that, as  $(x_\ell, y_\ell)$  is in  $(M \times M) \setminus (V \times V)$ , at least one of the points  $x_\ell$  or  $y_\ell$  must be contained in  $M \setminus V$ . Therefore, the subset  $M \setminus V$  contains an infinite subsequence of at least one of the sequences  $(x_\ell)$  and  $(y_\ell)$ . As both sequences converge to  $x_0$  and  $M \setminus V$  is closed, we then deduce that  $x_0$  must be contained in  $M \setminus V$ , and hence  $(x_0, \xi_0)$  is contained in  $GM|_{M \setminus V}$ . It follows from this that  $\mathcal{G} \setminus \mathcal{V}$  is a closed subset of  $\mathcal{G}$ , and hence  $\mathcal{V}$  is an open subset.

Let us prove that  $\gamma_\kappa$  is a homeomorphism. It is straightforward to check that  $\gamma_\kappa$  is a bijection whose inverse map is given by (13.9). Moreover, it is immediate from the definition of the topology of  $\mathcal{G}$  that  $\gamma_\kappa$  is a continuous map. Therefore, we only need to check that the inverse map  $\gamma_\kappa^{-1}$  is continuous. It is clear that it is continuous on  $\mathcal{U}_0 = \{(x, y, t) \in \mathcal{U}; t > 0\}$  and on  $U \times \mathbb{R}^n \times \{0\}$ . In addition, let  $((x_\ell, y_\ell, t_\ell))_{\ell \geq 1}$  be a sequence in  $\mathcal{U}_0$  converging to a boundary point  $(x_0, y_0, 0) \in U \times \mathbb{R}^n \times \{0\}$ . For  $\ell = 0, 1, \dots$ , set  $\hat{x}_\ell = \kappa^{-1}(x_\ell)$  and  $\hat{y}_\ell = (\varepsilon_{x_\ell}^\kappa \circ \kappa)^{-1}(t_\ell \cdot y_\ell)$ , so that  $\gamma_\kappa^{-1}(x_\ell, y_\ell, t_\ell) = (\hat{x}_\ell, \hat{y}_\ell, t_\ell)$ . As  $\ell \rightarrow \infty$ , we have

$$\hat{x}_\ell \longrightarrow \kappa^{-1}(x_0), \quad \hat{y}_\ell \longrightarrow (\varepsilon_{x_0}^\kappa \circ \kappa)^{-1}(0) = \kappa(x_0), \quad t_\ell \longrightarrow 0.$$

Moreover, we have

$$t_\ell \cdot [(\varepsilon_{\kappa(\hat{x}_\ell)} \circ \kappa)(\hat{y}_\ell)] = y_\ell \longrightarrow y_0 = \kappa'_H(\kappa^{-1}(x_0)) [(\kappa^{-1})'_H(x_0)y_0].$$

Therefore, we see that in  $\mathcal{G}$  we have

$$\gamma_\kappa^{-1}(x_\ell, y_\ell, t_\ell) \longrightarrow (\kappa^{-1}(x_0), (\kappa^{-1})'_H(x_0)y_0) = \gamma_\kappa^{-1}(x_0, y_0, 0).$$

It follows from all this that  $\gamma_\kappa^{-1}$  is continuous, and so  $\gamma_\kappa$  is a homeomorphism. The proof is complete.  $\square$

**Lemma 13.15.** *Let  $\kappa_1$  be another local  $H$ -chart with domain  $V_1$  and range  $U_1$ . In addition, let  $\gamma_{\kappa_1} : \mathcal{V}_1 \rightarrow \mathcal{U}_1$  be the homeomorphism (13.8) associated with  $\kappa_1$ . Then the transition map  $\gamma_{\kappa_1} \circ \gamma_\kappa^{-1} : \mathcal{U} \cap \gamma_\kappa(\mathcal{V}) \rightarrow \gamma_{\kappa_1}(\mathcal{V}) \cap \mathcal{U}_1$  is a smooth diffeomorphism between open subsets of  $\mathbb{R}^n \times \mathbb{R}^n \times [0, \infty)$ .*

*Proof.* We note that the inverse map of  $\gamma_{\kappa_1} \circ \gamma_\kappa^{-1}$  is  $\gamma_\kappa \circ \gamma_{\kappa_1}^{-1} : \mathcal{U}_1 \cap \gamma_{\kappa_1}(\mathcal{V}) \rightarrow \gamma_\kappa(\mathcal{V}) \cap \mathcal{U}$ . Therefore, it is enough to prove that  $\gamma_{\kappa_1} \circ \gamma_\kappa^{-1}$  is a smooth map, since swapping the roles of  $\kappa$  and  $\kappa_1$  shows that its inverse map is smooth as well.

Let  $\phi = \kappa_1 \circ \kappa^{-1}$  be the corresponding transition map with domain  $U_\phi := U \cap \kappa(V_1)$ . In addition, let  $(X_1, \dots, X_n)$  (resp.,  $(X'_1, \dots, X'_n)$ ) be the  $H$ -frame associated with  $\kappa$  (resp.,  $\kappa'$ ) in



the local coordinates defined by  $\kappa$  (resp.,  $\kappa_1$ ). We observe that a point  $(x, y, t) \in U \times \mathbb{R}^n \times (0, \infty)$  is contained in  $\mathcal{U} \cap \gamma_\kappa(\mathcal{V}_1)$  if and only if

$$\kappa^{-1}(x) \in V_1 \quad \text{and} \quad (\varepsilon_x^X)^{-1}(t \cdot y) \in U \quad \text{and} \quad \kappa^{-1} \circ (\varepsilon_x^X)^{-1}(t \cdot y) \in V_1.$$

It then follows that  $\mathcal{U} \cap \gamma_\kappa(\mathcal{V}_1)$  agrees with

$$\mathcal{U}_\phi := \{(x, y, t) \in U_\phi \times \mathbb{R}^n \times [0, \infty); (\varepsilon_x^X)^{-1}(t \cdot y) \in U_\phi\}.$$

Bearing this in mind, Proposition 11.12 ensures us there is a smooth map  $\Theta$  from  $\mathcal{U}_\phi = \mathcal{U} \cap \gamma_\kappa(\mathcal{V}_1)$  to  $\mathbb{R}^n$  such that

$$t^{-1} \cdot \left[ (\varepsilon_{\phi(x)}^{X'} \circ \phi \circ (\varepsilon_x^X)^{-1})(t \cdot y) \right] = \phi'_H(x)y + t\Theta(x, y, t) \quad \text{for all } (x, y, t) \in \mathcal{U} \cap \gamma_\kappa(\mathcal{V}_1).$$

Let  $(x, y, t) \in \mathcal{U} \cap \gamma_\kappa(\mathcal{V}_1)$ . If  $t > 0$ , then

$$(13.10) \quad \begin{aligned} \gamma_{\kappa_1} \circ \gamma_\kappa^{-1}(x, y, t) &= \left( \kappa_1 [\kappa^{-1}(x)], t^{-1} \cdot \left[ (\varepsilon_{\kappa_1[\kappa^{-1}(x)]}^{\kappa_1} \circ \kappa_1) \circ (\varepsilon_x^\kappa \circ \kappa)^{-1}(t \cdot y) \right], t \right) \\ &= \left( \phi(x), t^{-1} \cdot \left[ (\varepsilon_{\phi(x)}^{X'} \circ \phi \circ (\varepsilon_x^X)^{-1})(t \cdot y) \right], t \right). \end{aligned}$$

If  $t = 0$ , then using Proposition 2.30 we obtain

$$\gamma_{\kappa_1} \circ \gamma_\kappa^{-1}(x, y, 0) = (\kappa_1 [\kappa^{-1}(x)], (\kappa_1)'_H(\kappa(x))(\kappa^{-1})'_H(x)y, 0) = (\phi(x), \phi'_H(x)y, 0).$$

Combining this with (13.10) we deduce that, for all  $(x, y, t) \in \mathcal{U} \cap \gamma_\kappa(\mathcal{V}_1)$ , we have

$$\gamma_{\kappa_1} \circ \gamma_\kappa^{-1}(x, y, t) = (\phi(x), \phi'_H(x)y + t\Theta(x, y, t), t).$$

This shows that the transition map  $\gamma_{\kappa_1} \circ \gamma_\kappa^{-1}$  is smooth. The proof is thus complete.  $\square$

Lemma 13.14 and Lemma 13.15 precisely show that, as  $\kappa$  ranges over local  $H$ -charts, the maps  $\gamma_\kappa$  form a system of local charts near  $GM$ . This leads us to the following definition.

**Definition 13.16.** The differentiable structure of  $\mathcal{G}$  is the unique differentiable structure such that

- The inclusion of  $\mathcal{G}_+ = M \times M \times (0, \infty)$  into  $\mathcal{G}$  is a smooth embedding.
- A system of local charts near  $GM$  is given by the maps  $\gamma_\kappa$ , where  $\kappa$  ranges over all local  $H$ -charts on  $M$ .

The above differentiable structure turns  $\mathcal{G}$  into a smooth manifold with boundary whose interior is  $\mathcal{G}_+ = M \times M \times (0, \infty)$  and its boundary is  $GM$ .

**Lemma 13.17.** *The unit map  $\epsilon : \mathcal{G}^{(0)} \rightarrow \mathcal{G}$  is a smooth embedding.*

*Proof.* As the unit map  $\epsilon$  is one-to-one we only have to show it is an immersion. Let  $\kappa$  be a local  $H$ -chart with domain  $V$  and range  $U$  and  $\gamma_\kappa : \mathcal{V} \rightarrow \mathcal{U}$  the local chart (13.8) associated with  $\kappa$ . Let  $(x, t) \in U \times [0, \infty)$ . If  $t > 0$ , then

$$(\gamma_\kappa \circ \epsilon)(\kappa(x), t) = \gamma_\kappa(\kappa^{-1}(x), \kappa^{-1}(x), t) = (x, t \cdot \varepsilon_x^\kappa(x), t) = (x, 0, t).$$

If  $t = 0$ , then  $(\gamma_\kappa \circ \epsilon)(\kappa(x), 0) = \gamma_\kappa(\kappa^{-1}(x), 0) = (x, 0, 0)$ . Thus,

$$(\gamma_\kappa \circ \epsilon)(\kappa(x), t) = (x, 0, t) \quad \text{for all } (x, t) \in U \times [0, \infty).$$

It follows from this that the unit map  $\epsilon$  is an immersion. The proof is complete.  $\square$

**Lemma 13.18.** *The range and source maps  $r, s : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$  are submersions.*

*Proof.* It is immediate from their definitions that the range and source maps are submersions on  $\mathcal{G}^+$ . Therefore, we only need to check there are submersions near the boundary  $GM$ . Let  $\kappa$  be a local  $H$ -chart with domain  $V$  and range  $U$  and let  $\gamma_\kappa : \mathcal{V} \rightarrow \mathcal{U}$  be the local chart (13.8) associated with  $\kappa$ . Let  $(x, y, t) \in \mathcal{U}$ . If  $t > 0$ , then

$$\begin{aligned} (r \circ \gamma_\kappa^{-1})(x, y, t) &= r(\kappa^{-1}(x), (\varepsilon_x^\kappa \circ \kappa)^{-1}(t \cdot y), t) = (\kappa^{-1}(x), t), \\ (s \circ \gamma_\kappa^{-1})(x, y, t) &= s(\kappa^{-1}(x), (\varepsilon_x^\kappa \circ \kappa)^{-1}(t \cdot y), t) = ((\varepsilon_x^\kappa \circ \kappa)^{-1}(t \cdot y), t). \end{aligned}$$

If  $t = 0$ , then  $\gamma_\kappa^{-1}(x, y, 0) = (\kappa^{-1}(x), (\kappa^{-1})'_H(x)y)$ , and so we have

$$(r \circ \gamma_\kappa^{-1})(x, y, 0) = (s \circ \gamma_\kappa^{-1})(x, y, 0) = (\kappa^{-1}(x), 0).$$

Note that  $(\varepsilon_x^\kappa \circ \kappa)^{-1}(0) = \kappa^{-1} \circ (\varepsilon_x^\kappa)^{-1}(0) = \kappa^{-1}(x)$ . Therefore, for all  $(x, y, t) \in \mathcal{U}$ , we have

$$((\kappa \otimes \text{id}) \circ r \circ \gamma_\kappa^{-1})(x, y, t) = (x, t) \quad \text{and} \quad ((\kappa \otimes \text{id}) \circ s \circ \gamma_\kappa^{-1})(x, y, t) = ((\varepsilon_x^\kappa)^{-1}(t \cdot y), t).$$

It immediately follows from this that the range map  $r$  is a submersion near  $GM$ . Moreover, as the normal Carnot coordinate maps are diffeomorphisms, we also see that the source map  $s$  is a submersion near  $GM$  too. The proof is complete.  $\square$

**Lemma 13.19.** *The inverse map  $\iota : \mathcal{G} \rightarrow \mathcal{G}$  is a diffeomorphism.*

*Proof.* As the inverse map is an involution and restricts to a smooth map on  $\mathcal{G}_+ = M \times M \times (0, \infty)$ , we only have to check it is smooth near the boundary  $GM$ . Let  $\kappa$  be a local  $H$ -chart with domain  $V$  and range  $U$  and  $\gamma_\kappa : \mathcal{V} \rightarrow \mathcal{U}$  the local chart (13.8) associated with  $\kappa$ . In addition, let  $U \times \mathbb{R}^n \ni (x, y) \rightarrow \varepsilon_x(y)$  the normal Carnot coordinate map associated with  $\kappa$ . We observe that  $\iota(\mathcal{V}) = \mathcal{V}$ . Set  $\iota_\kappa = \gamma_\kappa \circ \iota \circ \gamma_\kappa^{-1}$ ; this is a map from  $\mathcal{U}$  to itself. Let  $(x, y, t) \in \mathcal{U}$ . If  $t > 0$ , then

$$(13.11) \quad \iota_\kappa(x, y, t) = \gamma_\kappa((\varepsilon_x \circ \kappa)^{-1}(t \cdot y), \kappa^{-1}(x), t) = (\varepsilon_x^{-1}(t \cdot y), t^{-1} \cdot \varepsilon_{\varepsilon_x^{-1}(t \cdot y)}(x), t).$$

If  $t = 0$ , then using Proposition 2.31 we obtain

$$(13.12) \quad \iota_\kappa(x, y, t) = \gamma_\kappa(\kappa^{-1}(x), -(\kappa^{-1})'_H(x)y) = \gamma_\kappa(\kappa^{-1}(x), [\kappa'_H(\kappa^{-1}(x))]^{-1}(-y)) = (x, -y).$$

For  $(x, y, t) \in \mathcal{U}$  set  $\Theta(x, y, t) = t^{-1} \cdot \varepsilon_{\varepsilon_x^{-1}(t \cdot y)}(x) + t \cdot y$ . Then  $\Theta$  is a smooth map from  $\mathcal{U}$  to  $\mathbb{R}^n$ . Let  $x \in U$ . Applying Proposition 12.3 to  $\phi = \varepsilon_x$  and using Lemma 8.3 we see that, for all  $(y, z) \in \mathbb{R}^n \times \mathbb{R}^n$  and as  $t \rightarrow 0^+$ , we have

$$t^{-1} \cdot [\varepsilon_{\varepsilon_x^{-1}(t \cdot y)} \circ \varepsilon_x^{-1}(t \cdot y)] = (-y) \cdot z + O(t).$$

As  $(\varepsilon_x)^{-1}(0) = x$ , for  $z = 0$  this gives  $t^{-1} \cdot [\varepsilon_{\varepsilon_x^{-1}(t \cdot y)}(x)] = -y + O(t)$ , i.e.,  $t^{-1} \cdot \Theta(x, y, t) = O(t)$  as  $t \rightarrow 0^+$ . Using Lemma 8.1 we then deduce there is a smooth map  $\tilde{\Theta} : \mathcal{U} \rightarrow \mathbb{R}^n$  such that

$$t^{-1} \cdot \varepsilon_{\varepsilon_x^{-1}(t \cdot y)}(x) + y = t^{-1} \cdot \Theta(x, y, t) = t\tilde{\Theta}(x, y, t) \quad \text{for all } (x, y, t) \in \mathcal{U}.$$

Combining this with (13.11)–(13.12) we see that

$$\iota_\kappa(x, y, t) = (\varepsilon_x^{-1}(t \cdot y), -y + t\tilde{\Theta}(x, y, t), t) \quad \text{for all } (x, y, t) \in \mathcal{U}.$$

We then deduce that  $\iota_\kappa$  is a smooth map from  $\mathcal{U}$  to itself. This shows that the inverse map is smooth near the boundary  $GM$ . The proof is complete.  $\square$

**Lemma 13.20.** *The multiplication map  $\mu : \mathcal{G}^{(2)} \rightarrow \mathcal{G}$  is smooth.*

*Proof.* It follows from Lemma 13.18 that  $\mathcal{G}^{(2)}$  is a manifold with boundary. It will be convenient to introduce the following notation:

$$\begin{aligned} [x, y, z, t] &= ((x, y, t), (y, z, t)) \in \mathcal{G}_+ \times \mathcal{G}_+ \quad \text{for } x, y, z \in M \text{ and } t > 0, \\ [x, \xi, \eta] &= ((x, \xi), (x, \eta)) \in GM \times GM \quad \text{for } x \in M \text{ and } \xi, \eta \in G_x M. \end{aligned}$$

Using this notation, we have  $\mathcal{G}^{(2)} = GM^{(2)} \sqcup \mathcal{G}_+^{(2)}$ , with

$$\mathcal{G}_+^{(2)} := \{[x, y, z, t]; x, y, z \in M, t > 0\} \quad \text{and} \quad GM^{(2)} = \{[x, \xi, \eta]; x \in M, \xi, \eta \in G_x M\}.$$

The interior  $\mathcal{G}_+^{(2)}$  is a codimension 2 submanifold of  $\mathcal{G}_+ \times \mathcal{G}_+$  which is diffeomorphic to  $M \times M \times M \times (0, \infty)$ . The boundary  $GM^{(2)}$  is an hypersurface in  $GM \times GM$ .

It is immediate that the multiplication map induces a smooth map from  $\mathcal{G}_+^{(2)}$  to  $\mathcal{G}_+$ . Therefore, in order to prove that  $\mu$  is a smooth map from  $\mathcal{G}^{(2)}$  to  $\mathcal{G}$  we only have to check it is smooth near the boundary  $GM^{(2)}$ . Bearing this in mind, a system of local charts near  $GM^{(2)}$  is obtained as follows.

Let  $\kappa$  be a local  $H$ -chart with domain  $V$  and range  $U$ . Let  $\gamma_\kappa : \mathcal{V} \rightarrow \mathcal{U}$  be the local chart (13.8) of  $\mathcal{G}$  associated with  $\kappa$ . In addition, we let  $U \times \mathbb{R}^n \ni (x, y) \rightarrow \varepsilon_x(y)$  be the normal Carnot coordinate map associated with  $\kappa$ . Set

$$\mathcal{U}^{(2)} = \left\{ (x, y, z, t) \in U \times \mathbb{R}^n \times \mathbb{R}^n \times [0, \infty); \varepsilon_x^{-1}(t \cdot y) \in U \text{ and } (\varepsilon_{\varepsilon_x^{-1}(t \cdot y)})^{-1}(t \cdot z) \in U \right\}.$$

We also set  $\mathcal{V}^{(2)} = GM|_V^{(2)} \sqcup \mathcal{V}_+^{(2)}$ , with

$$\mathcal{V}_+^{(2)} := \{[x, y, z, t]; x, y, z \in V, t > 0\} \quad \text{and} \quad GM|_V^{(2)} = \{[x, \xi, \eta]; x \in V, \xi, \eta \in G_x M\}.$$

We then define the map  $\gamma_\kappa^{(2)} : \mathcal{V}^{(2)} \rightarrow \mathcal{U}^{(2)}$  by letting

$$\begin{aligned} \gamma_\kappa^{(2)}([x, y, z, t]) &= (\kappa(x), t^{-1} \cdot [(\varepsilon_{\kappa(x)} \circ \kappa)(y)], t^{-1} \cdot [(\varepsilon_{\kappa(y)} \circ \kappa)(y)]) \quad \text{for } x, y, z \in V \text{ and } t > 0, \\ \gamma_\kappa^{(2)}([x, \xi, \eta]) &= (\kappa(x), (\kappa^{-1})'_H(x)\xi, (\kappa^{-1})'_H(x)\eta) \quad \text{for } x \in V \text{ and } \xi, \eta \in G_x M. \end{aligned}$$

The inverse map  $(\gamma_\kappa^{(2)})^{-1} : \mathcal{U}^{(2)} \rightarrow \mathcal{V}^{(2)}$  is such that, for all  $(x, y, z, t) \in \mathcal{U}^{(2)}$ , we have

$$(\gamma_\kappa^{(2)})^{-1}(x, y, z, t) = \begin{cases} \left[ \kappa^{-1}(x), (\varepsilon_x \circ \kappa)^{-1}(t \cdot y), (\varepsilon_{\varepsilon_x^{-1}(t \cdot y)} \circ \kappa)^{-1}(t \cdot z) \right] & \text{if } t > 0, \\ \left[ \kappa^{-1}(x), (\kappa^{-1})'_H(x)y, (\kappa^{-1})'_H(x)z \right] & \text{if } t = 0. \end{cases}$$

As  $\kappa$  ranges over local  $H$ -charts, the maps  $\gamma_\kappa^{(2)}$  form a system of local charts near the boundary  $GM^{(2)}$  of  $\mathcal{G}^{(2)}$ .

Keeping on using the notation above, set  $\mu_\kappa = \gamma_\kappa \circ \mu \circ (\gamma_\kappa^{(2)})^{-1}$ ; this is a map from  $\mathcal{U}^{(2)}$  to  $\mathcal{U}$ . Let  $(x, y, z, t) \in \mathcal{U}^{(2)}$ . If  $t > 0$ , then

$$(13.13) \quad \mu_\kappa(x, y, z, t) = \gamma_\kappa \left( \kappa^{-1}(x), (\varepsilon_{\varepsilon_x^{-1}(t \cdot y)} \circ \kappa)^{-1}(t \cdot z), t \right) = \left( x, t^{-1} \cdot [(\varepsilon_x \circ \varepsilon_{\varepsilon_x^{-1}(t \cdot y)}^{-1})(t \cdot z)], t \right).$$

If  $t = 0$ , then using Proposition 2.31 we get

$$\begin{aligned} (13.14) \quad \mu_\kappa(x, y, z, 0) &= \gamma_\kappa \left( \kappa^{-1}(x), [(\kappa^{-1})'_H(x)y], [(\kappa^{-1})'_H(x)z] \right) \\ &= \gamma_\kappa \left( \kappa^{-1}(x), \kappa'_H(\kappa^{-1}(x))^{-1}(y \cdot z) \right) \\ &= (x, y \cdot z). \end{aligned}$$

*Claim.* There is a smooth map  $\Theta : \mathcal{U}^{(2)} \rightarrow \mathbb{R}^n$  such that

$$t^{-1} \cdot [(\varepsilon_x \circ \varepsilon_{\varepsilon_x^{-1}(t \cdot y)}^{-1})(t \cdot z)] = y \cdot z + t\Theta(x, y, z, t) \quad \text{for all } (x, y, z, t) \in \mathcal{U}^{(2)}.$$

*Proof of the Claim.* For  $(x, y, z, t) \in \mathcal{U}$  set

$$\tilde{\Theta}(x, y, z, t) = (\varepsilon_x \circ \varepsilon_{\varepsilon_x^{-1}(t \cdot y)}^{-1})(t \cdot z) - t \cdot (y \cdot z).$$

Then  $\tilde{\Theta}$  is a smooth map from  $\mathcal{U}$  to  $\mathbb{R}^n$ . Let  $(X_1, \dots, X_n)$  be the  $H$ -frame associated with  $\kappa$  in the local coordinates defined by  $\kappa$ , so that  $\varepsilon_x = \varepsilon_x^\kappa = \varepsilon_x^X$ . Let  $(x, y, z, t) \in \mathcal{U}$ . Using Proposition 11.16 we obtain

$$\begin{aligned} t^{-1} \cdot [(\varepsilon_x \circ \varepsilon_{\varepsilon_x^{-1}(t \cdot y)}^{-1})(t \cdot z)] &= \left( \delta_t^{-1} \circ \varepsilon_{t \cdot y}^{(\varepsilon_x)^* X} \circ \delta_t \right)^{-1} \circ \delta_t^{-1} \circ \varepsilon_{t \cdot y}^{(\varepsilon_x)^* X} \circ \varepsilon_x \circ \varepsilon_{\varepsilon_x^{-1}(t \cdot y)}^{-1}(t \cdot z) \\ &= \left( \varepsilon_y^{t \cdot \delta_t^*(\varepsilon_x)^* X} \right)^{-1} \left[ t^{-1} \cdot \left( \varepsilon_{t \cdot y}^{(\varepsilon_x)^* X} \circ \varepsilon_x \circ \varepsilon_{\varepsilon_x^{-1}(t \cdot y)}^{-1} \right)(t \cdot z) \right]. \end{aligned}$$

Let  $x \in U$  and  $y \in \mathbb{R}^n$ . As  $\varepsilon_x = \varepsilon_x^X$  provides us with Carnot coordinates at  $x$  adapted to the  $H$ -frame  $(X_1, \dots, X_n)$ , we know that, for  $j = 1, \dots, n$ , we have

$$t^{w_j} \delta_t^*(\varepsilon_x)^* X_j = X_j^x + O(t) \quad \text{as } t \rightarrow 0^+.$$

Therefore, using Proposition 10.4 we deduce that, as  $t \rightarrow 0^+$  and uniformly on compact subsets of  $\mathbb{R}^n$ , we have

$$(13.15) \quad \left( \varepsilon_y^{t \cdot \delta_t^*(\varepsilon_x)^* X} \right)^{-1}(z) = \left( \varepsilon_y^{X^x} \right)^{-1}(z) + O(t).$$

Let  $x \in U$  and set  $\mathcal{W} = U \times \mathbb{R}^n \times [0, \infty)$ . Applying Proposition 11.12 to  $\phi = \varepsilon_x$  and  $X'_j = (\varepsilon_x)_* X_j$ ,  $j = 1, \dots, n$ , and using Lemma 11.1 shows there is a smooth map  $\tilde{\Theta} : \mathcal{W} \rightarrow \mathbb{R}^n$  such that

$$(13.16) \quad t^{-1} \cdot \left( \varepsilon_{\varepsilon_x(y)}^{(\varepsilon_x)_* X} \circ \varepsilon_x \circ \varepsilon_y^{-1} \right) (t \cdot z) = z + t\tilde{\Theta}(y, z, t) \quad \text{for all } (y, z, t) \in \mathcal{W}.$$

Let  $(y, z) \in \mathbb{R}^n \times \mathbb{R}^n$ . Substituting  $\varepsilon_x^{-1}(t \cdot y)$  for  $y$  in (13.16) with  $t$  small enough, we see that, as  $t \rightarrow 0^+$ , we have

$$(13.17) \quad t^{-1} \cdot \left( \varepsilon_{\varepsilon_x^{-1}(t \cdot y)}^{(\varepsilon_x)_* X} \circ \varepsilon_x \circ \varepsilon_{\varepsilon_x^{-1}(t \cdot y)}^{-1} \right) (t \cdot z) = z + t\tilde{\Theta}(\varepsilon_x^{-1}(t \cdot y), z, t) = z + O(t).$$

Combining this with (13.15) and (13.17) gives

$$t^{-1} \cdot \left[ (\varepsilon_x \circ \varepsilon_{\varepsilon_x^{-1}(t \cdot y)}^{-1})(t \cdot z) \right] = y \cdot (z + O(t)) + O(t) = y \cdot z + O(t).$$

This shows that, for all  $(x, y, z) \in U \times \mathbb{R}^n \times \mathbb{R}^n$  and as  $t \rightarrow 0^+$ , we have  $t^{-1} \cdot \tilde{\Theta}(x, y, z, t) = O(t)$ . Using Lemma 12.2 we then deduce there is a smooth map  $\Theta : \mathcal{U} \rightarrow \mathbb{R}^n$  such that

$$t^{-1} \cdot \left[ (\varepsilon_x \circ \varepsilon_{\varepsilon_x^{-1}(t \cdot y)}^{-1})(t \cdot z) \right] - y \cdot z = t^{-1} \cdot \tilde{\Theta}(x, y, z, t) = t\Theta(x, y, z, t).$$

This proves the claim.  $\square$

Combining the above claim with (13.13)–(13.14) we see that

$$\mu_\kappa(x, y, z, t) = (x, y \cdot z + t\Theta(x, y, z, t), t) \quad \text{for all } (x, y, z, t) \in \mathcal{U}^{(2)}.$$

As  $\Theta$  is a smooth map, we deduce that  $\mu_\kappa$  is a smooth map from  $\mathcal{U}^2$  to  $\mathcal{U}$ . As the maps  $\gamma_\kappa^{(2)}$  form a system of local charts near the boundary  $GM^{(2)}$ , it then follows that the multiplication map is smooth near  $GM^{(2)}$ . This completes the proof.  $\square$

Combining Lemmas 13.17 through 13.20 we then arrive at the main result of this section.

**Theorem 13.21.** *The groupoid  $\mathcal{G} = \mathcal{G}_H M$  is a b-differentiable groupoid.*

*Remark 13.22.* When  $r = 1$  the groupoid  $\mathcal{G}_H M$  agrees with Connes' tangent groupoid (13.2). Incidentally, in this case Theorem 13.21 reduces to Proposition 13.10.

*Remark 13.23.* In the case of a Heisenberg manifold  $(M, H)$ , i.e., when  $r = 2$  and  $\text{corank}(H_1) = 1$ , we recover the results of [Po1] (see also [vE1]). We also refer to [JvE] for an alternative construction of the tangent groupoid in the case of step 2 Carnot manifolds.

**13.4. Functoriality.** Let us now look at the functoriality of the construction of the tangent groupoid of a Carnot manifold. Let  $(M, H)$  and  $(M', H')$  be Carnot manifolds of step  $r$  with respective tangent groupoids  $\mathcal{G} = \mathcal{G}_H M$  and  $\mathcal{G}' = \mathcal{G}_{H'} M'$ . (We do not assume that  $M$  and  $M'$  have same dimension.) Given a Carnot map  $\phi : M \rightarrow M'$  we define maps  $\Phi : \mathcal{G} \rightarrow \mathcal{G}'$  and  $\Phi^0 : \mathcal{G}^0 \rightarrow (\mathcal{G}')^0$  by

$$(13.18) \quad \Phi(x, y, t) = (\phi(x), \phi(y), t) \quad \text{for } t > 0 \text{ and } x, y \in M,$$

$$(13.19) \quad \Phi(x, \xi) = (\phi(x), \phi'_H(x)\xi) \quad \text{for } (x, \xi) \in GM,$$

$$(13.20) \quad \Phi^0(x, t) = (\phi(x), t) \quad \text{for } t \geq 0 \text{ and } x \in M.$$

The pair  $(\Phi, \Phi^0)$  provides us with a morphism of groupoids in the sense mentioned in Remark 13.3.

**Lemma 13.24.** *The maps  $\Phi$  and  $\Phi^0$  are smooth maps between manifolds with boundary.*

*Proof.* It is immediate that  $\Phi^0$  is a smooth map between the manifolds with boundary  $\mathcal{G}^0 = M \times [0, \infty)$  and  $(\mathcal{G}')^0 = M' \times [0, \infty)$ . It is also immediate that  $\Phi$  induces a smooth map between the interiors  $\mathcal{G}_+ = M \times M \times (0, \infty)$  and  $\mathcal{G}'_+ = M' \times M' \times (0, \infty)$ . Therefore, we only check that  $\Phi$  is smooth near the boundary  $GM$ . Given  $a \in M$  and setting  $a' = \phi(a)$ , let  $\kappa_a$  (resp.,  $\kappa_{a'}$ ) be a local  $H$ -chart (resp.,  $H'$ -chart) near  $a$  (resp.,  $a'$ ) with domain  $V$  (resp.,  $V'$ ) and range  $U$  (resp.,  $U'$ ). Without any loss of generality we may assume that  $\phi(V)$  is contained in  $V'$ . We also let  $\gamma_{\kappa_a} : \mathcal{V} \rightarrow \mathcal{U}$  (resp.,  $\gamma_{\kappa_{a'}} : \mathcal{V}' \rightarrow \mathcal{U}'$ ) be the local chart (13.8) associated with  $\kappa_a$  (resp.,  $\kappa_{a'}$ ). Note that  $\mathcal{U}$  and  $\mathcal{U}'$  are open subsets of  $\mathbb{R}^n \times \mathbb{R}^n \times [0, \infty)$  and  $\mathbb{R}^{n'} \times \mathbb{R}^{n'} \times [0, \infty)$ , respectively, where

$n$  and  $n'$  are the respective dimensions of  $M$  and  $M'$ . In addition, let  $U \times \mathbb{R}^n \ni (x, y) \rightarrow \varepsilon_x^a(y)$  and  $U' \times \mathbb{R}^{n'} \ni (x, y) \rightarrow \varepsilon_x^{a'}(y)$  be the respective normal Carnot coordinate map associated with  $\kappa_a$  and  $\kappa_{a'}$ .

Set  $\Phi_\kappa = \gamma_{\kappa_{a'}} \circ \Phi \circ \gamma_{\kappa_a}^{-1} : \mathcal{U} \rightarrow \mathcal{U}'$ . Let  $(x, y, t) \in \mathcal{U}$ . If  $t > 0$ , then we have

$$\begin{aligned}
 \Phi_\kappa(x, y, t) &= \gamma_{\kappa_{a'}} \circ \Phi \left( \kappa_a^{-1}(x), (\varepsilon_x^a \circ \kappa_a)^{-1}(t \cdot x), t \right) \\
 (13.21) \quad &= \gamma_{\kappa_{a'}} \left( \phi \circ \kappa_a^{-1}(x), \phi \circ \kappa_a^{-1} \circ (\varepsilon_x^a)^{-1}(t \cdot x), t \right) \\
 &= \left( \kappa_{a'} \circ \phi \circ \kappa_a^{-1}(x), t^{-1} \cdot [\varepsilon_{\kappa_{a'} \circ \phi \circ \kappa_a^{-1}(x)}^{a'} \circ (\kappa_{a'} \circ \phi \circ \kappa_a^{-1}) \circ (\varepsilon_x^a)^{-1}(t \cdot x)], t \right).
 \end{aligned}$$

If  $t = 0$ , then using Proposition 2.30 we get

$$\begin{aligned}
 \Phi_\kappa(x, y, 0) &= \gamma_{\kappa_{a'}} \circ \Phi \left( \kappa_a^{-1}(x), (\kappa_a^{-1})'_H(x)y \right) \\
 (13.22) \quad &= \gamma_{\kappa_{a'}} \left( \phi \circ \kappa_a^{-1}(x), (\phi \circ \kappa_a^{-1})'_H(x)y \right) \\
 &= \left( \kappa_{a'} \circ \phi \circ \kappa_a^{-1}(x), (\kappa_{a'} \circ \phi \circ \kappa_a^{-1})'_H(x)y, 0 \right).
 \end{aligned}$$

Bearing this in mind, it follows from Proposition 11.12 there is a smooth map  $\Theta : \mathcal{U} \rightarrow \mathcal{U}'$  such that, for all  $(x, y, t) \in \mathcal{U}$ , we have

$$t^{-1} \cdot [\varepsilon_{\kappa_{a'} \circ \phi \circ \kappa_a^{-1}(x)}^{a'} \circ (\kappa_{a'} \circ \phi \circ \kappa_a^{-1}) \circ (\varepsilon_x^a)^{-1}(t \cdot x)] = (\kappa_{a'} \circ \phi \circ \kappa_a^{-1})'_H(x)y + t\Theta(x, y, t).$$

Combining this with (13.21)–(13.22) we see that, for all  $(x, y, t) \in \mathcal{U}$ , we have

$$\Phi_\kappa(x, y, t) = \left( \kappa_{a'} \circ \phi \circ \kappa_a^{-1}(x), (\kappa_{a'} \circ \phi \circ \kappa_a^{-1})'_H(x)y + t\Theta(x, y, t), t \right).$$

This shows that  $\Phi_\kappa$  is a smooth map from  $\mathcal{U}$  to  $\mathcal{U}'$ . It then follows that  $\Phi$  is smooth near the boundary  $GM$ . The proof is complete.  $\square$

**Lemma 13.25.** *Let  $(M'', H'')$  be a step  $r$  Carnot manifold with tangent groupoid  $\mathcal{G}'' = \mathcal{G}_{H''}M''$ . Let  $\psi : M' \rightarrow M''$  be a Carnot manifold map and set  $\pi = \psi \circ \phi$ . In addition, let  $\Psi : \mathcal{G}' \rightarrow \mathcal{G}''$  (resp.,  $\Pi : \mathcal{G} \rightarrow \mathcal{G}''$ ) and  $\Psi^0 : (\mathcal{G}')^0 \rightarrow (\mathcal{G}'')^0$  (resp.,  $\Pi^0 : \mathcal{G}^0 \rightarrow (\mathcal{G}'')^0$ ) be the maps (13.18)–(13.20) associated with  $\Psi$  (resp.,  $\Pi$ ). Then*

$$\Pi = \Psi \circ \Phi \quad \text{and} \quad \Pi^0 = \Psi^0 \circ \Phi^0.$$

*Proof.* It is immediate that  $\Pi^0 = \Psi^0 \circ \Phi^0$  and  $\Pi = \Psi \circ \Phi$  on  $\mathcal{G}_+ = M \times M \times (0, \infty)$ . Moreover, it follows from Proposition 2.30 that  $\Pi = \Psi \circ \Phi$  on  $GM$ . This proves the result.  $\square$

Lemma 13.24 shows that the pair  $(\Phi, \Phi^0)$  provides us with a morphism of  $b$ -differentiable groupoids. Combining this with Lemma 13.25 we then arrive at the following statement.

**Theorem 13.26.** *The assignment  $(M, H) \rightarrow \mathcal{G}_H M$  is a functor from the category of step  $r$  Carnot manifolds to the category of  $b$ -differentiable groupoids.*

*Remark 13.27.* Much like we can associate a Lie algebra with any Lie group, with any differentiable groupoid  $\mathcal{G}$  is associated a Lie algebroid (see, e.g., [Ma]). In particular, this is important in the context of the pseudodifferential calculus on groupoids (see, e.g., [ALN, DS, LMN, MP, NWX, Va]). Recall that a Lie algebroid structure on a vector bundle  $A$  over a manifold  $M$  is given by

- (i) A Lie bracket on the space of sections  $C^\infty(M, A)$ .
- (ii) An anchor map  $\rho : A \rightarrow TM$ , i.e., a vector bundle map satisfying Leibniz's rule.

For a differentiable groupoid  $\mathcal{G}$  the associated Lie algebroid  $A\mathcal{G}$  is a vector bundle over the unit space  $M = \mathcal{G}^{(0)}$  and the anchor map is given the differential of the source map. In the case of the tangent groupoid  $\mathcal{G}_H M$  of a Carnot manifold  $(M, H)$  we thus obtain a fibration of manifold with boundary  $A\mathcal{G}_H M \rightarrow M \times [0, \infty)$ , where

$$A\mathcal{G}_H M = \mathfrak{g}M \bigsqcup \pi^*TM.$$

Here  $\mathfrak{g}M$  is the tangent Lie algebra bundle of  $(M, H)$  and  $\pi^*TM$  is the pullback of  $TM$  by the first factor projection  $M \times (0, \infty) \rightarrow M$ .

*Remark 13.28.* It would be very interesting to extend the construction of the tangent groupoid of a Carnot manifold to the setup of non-equiregular Carnot-Carathéodory manifolds. In this case the tangent group at a singular point  $a$  is the quotient of the Carnot manifold tangent group  $GM(a)$  by a finite group (see, e.g., [Be]). Therefore, we should expect the tangent groupoid to be a manifold whose boundary has the structure of an orbifold.

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